Zombie Lending and Policy Traps

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Abstract

We build a model with heterogeneous firms and banks to explain how accommodative policy can get trapped due to credit misallocation and its spillovers, as witnessed in Japan in the 1990s and in Europe in the 2010s. Monetary policy can implement efficient production following small negative shocks, but large shocks necessitate unconventional policy such as regulatory forbearance towards banks. Excessive accommodation, however, induces “diabolical sorting”, whereby low-capitalization banks lend to low-productivity “zombie” firms. Due to congestion externalities of zombie lending on healthier firms, policymakers avoiding short-term recessions can get trapped into protracted low rates, excessive forbearance, and permanent output losses.

Keywords: Credit misallocation, Evergreening, Forbearance, Bank capital, Conventional monetary policy, Unconventional monetary policy.


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1 Introduction

In this paper, we build a model with heterogeneous firms and banks to analyze how monetary and banking policies affect credit allocation and long-term economic outcomes. In particular, we explain why policy may get trapped into protracted low rates and excessive regulatory forbearance that are coincident with permanent output losses.

Since the housing and banking crisis in Japan in the early 1990s, regulatory forbearance towards banks has been increasingly used in conjunction with accommodative monetary policy in a bid to restore economic growth in the aftermath of aggregate shocks. This policy combination also found favor in the Eurozone following the global financial crisis of 2007–08 and especially after the European sovereign debt crisis in 2010–12. In both cases, despite an operative period exceeding its initial intentions and expectations, this policy script’s impact on economic growth has remained relatively muted. Starting with Peek and Rosengren (2005) and Caballero, Hoshi and Kashyap (2008), the literature has attributed this ineffectiveness of employed policies (at least in part) to credit misallocation and, in particular, to the phenomenon of zombie lending: the provision of subsidized credit to poorly performing firms by weakly capitalized banks.1

Figure 1 shows the incidence of this phenomenon, presenting the dynamic evolution of the share of “zombie firms” in Japan after its real estate crisis and in peripheral European countries (Italy, Spain, and Portugal) after the onset of the sovereign debt crisis. Furthermore, in all countries, the proliferation of zombie lending has been accompanied by adverse spillover effects to healthier firms in the economy, substantial and protracted slowdown of economic activity, and sluggish economic growth. Table 1 sheds light on these trends, reporting the changes in gross domestic product (GDP) growth and aggregate total factor productivity (TFP) growth in each country in the decade following the aggregate shocks. Japan experienced a reduction of 3.1 percentage points in GDP growth and 6.3 percentage points in aggregate TFP growth relative to the growth rates in the pre-crisis period. Moreover, the data also reveals a strong negative correlation between the change in the share of zombie firms and long-run TFP growth across industries. The peripheral European countries experienced similarly sharp GDP and TFP slowdowns in the post-sovereign crisis period when compared to Germany over the same period, and a similar negative correlation between the share of zombie firms and TFP growth at the industry level.

1Poorly-capitalized banks supply credit to poorly performing firms either because they are incentivized to engage in risk-shifting (gambling for resurrection) or because they attempt to avoid reporting losses on distressed positions, as documented in Giannetti and Simonov (2013), Acharya et al. 2019, Blattner et al. (2020), Gropp et al. 2020, Faria E Castro et al. 2021, Acharya et al. (2021a), and Schivardi et al. (2021), among others.
Interestingly, during these periods, policy makers reacted to the shocks by implementing a series of macro-financial measures, which featured timid capital injections into their national banking systems as well as incisive packages of forbearance measures in the form of implicit or explicit government guarantees, central bank liquidity support facilities, and delayed loss-recognition schemes. The former were not able to adequately recapitalize the financial intermediaries in the face of the shock.\(^2\) The latter helped lower banks’ cost of capital but also allowed weakly capitalized banks to extend new credit or evergreen existing loans to borrowers which should have otherwise been deemed insolvent.\(^3\)

Figure 1: Share of zombie firms in Japan and Europe before and after the aggregate shock

Note: On the y-axis, the figure reports the share of firms classified as zombie firms in Japan, Italy, Spain, and Portugal. The x-axis reports time (in years) since the aggregate shock hit the economy. For Japan, the aggregate shock is the burst of the real estate crisis (time zero is 1990); for Italy, Portugal, and Spain, the aggregate shock is the burst of the European sovereign debt crisis (time zero is 2010). The definition of zombie firm follows the one in Caballero et al. (2008). See Appendix A for information on the data sources.

Motivated by these facts, we build a tractable model that provides a unified framework simultaneously explaining and conforming to all of them. We start with a static setting that describes what happens within a period, before turning to the full dynamic model. The economy is populated by heterogeneous firms that differ in their productivity and risk. Firms’ investments require credit, which is provided by banks that are themselves heterogeneous in their level of capitalization. Banks face a portfolio problem, whose solution depends on their capital: they decide whether to invest in safe assets (meant to capture a wide range of non-loan assets, such as central bank reserves, government bonds, or safe

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\(^2\)See, for example, Peek and Rosengren (2005), Hoshi and Kashyap (2010), and Giannetti and Simonov (2013) for Japan, and Acharya et al. (2021a) for Europe.

\(^3\)In addition to references in footnote 2, Fukao (2004), Okamura (2011), and Storz et al. (2017) for Japan, and Acharya et al. 2019 and Blattner et al. (2020) for Europe.
Table 1: Zombie lending, aggregate outcomes, and bank capital in Japan and Europe

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<td>Italy</td>
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<td>Share of zombie firms</td>
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<td>7.7</td>
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<td>Annualized GDP growth</td>
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<td>-18.1</td>
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<td>Annualized aggregate TFP growth</td>
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<td>-10.7</td>
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<td>Banking system capitalization</td>
<td>-3.3</td>
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Note: The table reports the following statistics. The change (in percentage points) of the average share of zombie firms after the aggregate shock relative to the average share of zombie firms in the benchmark. The change (in percentage points) in the average capital ratio of the Japanese banking system relative to the average capital ratio in the benchmark. The change (in percentage points) in the annualized growth rate of real GDP growth and aggregate TFP relative to the annualized growth rates of GDP and aggregate TFP in the benchmark. For Japan, the aggregate shock is the burst of the real estate crisis in 1989–1990, the post-shock period is 1990–2001, and the benchmark is Japan itself in the pre-shock period (1986–1989). For Italy, Portugal, and Spain, the aggregate shock is the burst of the European sovereign debt crisis in 2010, the post-shock period is 2010–2019, and the benchmark is Germany in 2010–2019. The table also reports the correlation between the percentage change in an industry share of zombie firms in and the industry TFP growth rate. See Appendix A for additional details and information on the data sources.

Policies play a crucial role in banks’ incentives, and thereby the equilibrium allocation of credit. We summarize all the components of policy that affect bank decisions into two simple instruments: the risk-free rate \( R^f \) set by conventional monetary policy, and an “unconventional policy” or “forbearance” parameter \( p \), that determines the level of government guarantees granted to banks that are willing to lend. Accommodative conventional monetary policy makes lending more attractive by lowering the return on safe assets \( R^f \). This is a standard bank lending channel. Increasing forbearance also stimulates lending, by compressing the cost of funds associated with lending: a higher \( p \) lowers the cost of funds because a larger part of the bank loan risk is borne by the government. However, excessive forbearance can tilt banks’ portfolios towards riskier loans to less productive firms, which we refer to as the “zombie lending channel”.

mortgage-backed securities) or lend to the productive sector, and if so, to which type of firms.
The two-sided heterogeneity opens the door to the “diabolical sorting” documented in the data: banks with low capital and high leverage end up lending to less productive firms, even though aggregate output would be raised by letting these firms exit and be replaced by more productive entrants. The reason is that the subsidy from forbearance increases with the interaction of banks’ asset risk and leverage. This sorting between banks and firms leads to a delicate policy trade-off. While zombie lending and depressed creative destruction are the main perils on the side of poorly-capitalized banks, policymakers must also encourage well-capitalized banks to lend. The latter are not tempted by zombie lending, and may in fact invest in safe assets instead of lending to good firms. This tension—inducing well-capitalized banks to lend while preventing poorly-capitalized banks from engaging in zombie lending—is at the heart of our analysis of the optimal policy mix in response to exogenous shocks.

The output of good firms depends on an aggregate productivity or demand shock. Since zombie loans and safe assets are always less productive than loans to good firms, output reaches its potential if and only if all banks lend and there is no zombie lending. As long as the risk-free rate is not constrained, conventional monetary policy alone without any forbearance can achieve this objective. Without forbearance, there is no zombie lending by weak banks, while a sufficiently low risk-free rate encourages healthy banks to lend. Furthermore, larger negative shocks to fundamentals must be accommodated by a lower interest rate. Hence, if the shocks are large enough, conventional monetary policy runs into an effective lower bound on interest rates (ELB). This is where unconventional policy in the form of regulatory forbearance and its unintended consequences come into play.

Zombie lending reduces aggregate output and productivity because of the misallocation of credit and, as documented by in the empirical literature, because it can generate congestion externalities in input and output market markets that ultimately impair the productivity and growth prospects of healthy firms in the economy.4 We show that a small amount of forbearance is beneficial, as it can substitute for the constrained conventional monetary policy and help lower banks’ funding costs, thereby stimulating lending and output. Pushing on the forbearance string, however, will spur zombie lending by weak banks.

If policymakers put a low welfare weight on congestion externalities (for instance, because they are politically aligned with current labor or lobbied by bank shareholders), then

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4There is now a large body of evidence showing that the broad adverse effects of zombie lending on healthier firms translate into lower economic outcomes such as depressed aggregate employment, investment, productivity, innovation, and output (see the survey of Acharya et al. (2021b) and references therein). The empirical estimates available to date suggest that not only can the spillover effects be substantial, but also that the output losses due to spillover effects might be persistent and even compound over time.
they will focus on maximizing output by maximizing bank lending, ignoring concerns about the composition about lending. This is achieved by responding to larger shocks with more forbearance. However, if congestion externalities are large, we show that the optimal unconventional policy is non-monotonic in the size of the shocks: when shocks are moderate, forbearance should increase with the size of the shock as expected; but in the face of large shocks, policymakers should actually backtrack and reduce forbearance to avoid triggering zombie lending, even though this entails letting some banks invest in safe assets instead of lending. For large shocks, the output loss from zombie lending can far exceed the opportunity cost of not lending to some healthy firms. Thus, there exists a “reversal” level of unconventional policy, a counterpart to the “reversal interest rate” below which conventional monetary policy turns contractionary (Brunnermeier and Koby, 2019).

To further explore the intertemporal trade-offs posed by zombie lending, we expand on the static framework and model the dynamic interactions between policy actions, zombie lending, and aggregate outcomes. Zombie lending causes negative spillovers on the productivity of healthy firms in future period via the congestion externalities. The policy trade-off is thus dynamic, between maximizing short-run output and hurting future productivity. We show that the crucial parameter (corresponding to the welfare weight on congestion externalities in the static model) is the horizon of policymakers and consider two polar cases: “patient” policymakers, seeking to avoid future output losses; and, “myopic” policymakers willing to preserve incumbent firms at the expense of future productivity, due to capture, term limits, or reputational concerns that shorten their effective horizon.

In the main policy experiment we consider, the economy suffers a transitory exogenous shock to fundamentals, as in the static model. With a long policy horizon, the optimal response is exactly as in the static model: conventional monetary policy without forbearance achieves potential output for small shocks; some forbearance is optimal once the ELB binds when shocks remain moderate; and forbearance should decrease for large shocks to avoid any zombie lending and the associated congestion externalities. Myopic policymakers, on the other hand, implement the same joint policies for small and moderate shocks as patient policymakers, but respond very differently to large shocks. Since they are focused on the short-term benefits of zombie lending (e.g., avoiding a reallocation of labor), they keep increasing forbearance as shocks deepen.

The dynamic consequences of such myopic policy can be dire: we find that if spillovers from zombie lending to the productivity of healthy firms are strong enough, the optimal myopic policy response precipitates the economy into the following policy trap. Although
the exogenous shock is transitory, future policymakers face an endogenously low productivity due to the congestion externalities, and continue responding in the same accommodating way, keeping interest rates low and forbearance high. This keeps zombies alive, and productivity low, for at least another period. At the very least, this negative dynamic feedback generates endogenous persistence. In the extreme, for large enough initial shocks, the pattern repeats itself until the economy converges to a sclerosis steady state, defined as featuring a permanent combination of interest rates stuck at the ELB, high forbearance, zombie lending, and low output, reminiscent of the Japanese lost decades and the post-global-financial-crisis stagnation in the eurozone. In our theory, forward-looking policymakers should accept a “V-shaped” recession (i.e., sharp but transitory) precisely when fundamental shocks are large, which is exactly the opposite of what is argued in practice.5 We discuss two ways to exit the sclerosis steady state: a bank recapitalization, which improves banks’ incentives at some fiscal cost, and an improvement in productivity. Both need to be sufficiently strong, as timid interventions only have a transitory effect before the economy becomes trapped again.

The central role of bank capital in our analysis raises important questions: Do under-capitalized banks have incentives to issue more equity? And if not, can regulators eliminate the zombie lending problem by simply increasing capital requirements? We address these questions in an extension of our model, allowing for costly equity issuance, legacy lending, and capital requirements. We find that the same risk-shifting incentives affecting lending decisions also apply to capital structure decisions, preventing under-capitalized banks from raising enough capital to avoid zombie lending. Imposing high capital requirements can then deter zombie lending if the costs of breaking the relationship with a legacy zombie borrower (e.g., recognizing losses) are low enough. However, if these costs are high, we show that zombie lending becomes inevitable, in the sense that some banks will evergreen (lend to legacy zombie firms) for any level of capital requirement. Furthermore, there is a zombie-minimizing level of capital requirement and going beyond this level leads to even more zombie lending.

Overall, our model consistently explains a set of empirical facts connecting bank capitalization, credit misallocation, policy choices, and aggregate growth and productivity, following adverse economic shocks. In particular, it makes three important contributions.

5While this argument resembles some of the classic “liquidationist” views of, e.g., Hayek and Schumpeter, in our framework this conclusion is contingent on many factors such as the size of the shock, the policy space available to address the shock with conventional tools, and the state of capitalization of the banking sector when the shock hits.
First, it helps understand why in the face of large shocks, the policy response to re-
store economic growth may feature a combination of conventional policy in the form of
monetary accommodation and unconventional policy in the form of regulatory forbear-
ance towards banks. Unconventional policy arises in our model only when the conven-
tional policy hits an effective or zero lower bound, distinct from the modeling of regulatory
forbearance in the banking literature as arising from a time-inconsistency problem of reg-
ulation (Mailath and Mester, 1994).

Secondly, the model derives in equilibrium the empirically documented phenomenon
that regulatory forbearance leads to zombie lending and a diabolical sorting, whereby low-
capitalization banks extend new credit or evergreen existing loans to low-productivity
firms. It is this positive implication of the model that then allows for a meaningful norm-
mative analysis of the policies affecting bank incentives to engage in such lending.

Thirdly, by examining a dynamic setting in which zombie lending imposes congestion
externalities in the form of adverse productivity spillovers on healthier firms, the model
explains why economies facing large, but only transitory, shocks may jointly feature there-
after (i) a phase of delayed recovery and potentially permanent output losses, which we
call economic sclerosis; and, (ii) a policy trap whereby monetary accommodation and reg-
ulatory forbearance aimed at avoiding short-term recessions become entrenched even as
they persistently fail to restore long-term economic health. The possibility that a transi-
tory shock turns into permanent stagnation is the most salient feature of our analysis. A
key policy implication is that to avoid zombie lending and associated economic sclerosis,
it is important to maintain a well-capitalized banking system but also that bank capital
requirements need to be raised upfront rather than upon realization of economic shocks.

The remainder of the paper is organized as follows. After reviewing the related litera-
ture, we develop our baseline model in Section 2. In Section 3 we analyze optimal policy and
turn to the dynamic model in Section 4. Section 5 presents extensions around the role of
bank capital and capital requirements. Section 6 concludes with some directions for future
research.

Related Literature

Our model builds on the seminal contribution of Caballero et al. (2008) (henceforth CHK)
and extends it in two key dimensions. First, CHK features negative spillovers generated
by zombie firms due to congestion in input and output markets, but it does not explicitly
model the role of credit markets and financial intermediaries, and their incentives to extend
credit to low productivity firms. By contrast, the credit market, banks and their capital structures are front and center in our framework. Second, our model stresses the nexus between policy interventions, credit allocation, and aggregate outcomes. To the best of our knowledge, ours is the first theoretical treatment emphasizing the zombie lending-policy feedback loop, where macro-financial policies dynamically affect and are affected by banks’ credit supply choices. In this respect, our results on economic sclerosis and policy traps also speak to the stagnation traps analyzed by Benigno and Fornaro (2018), who highlight the ineffectiveness of conventional monetary policy alone in stimulating the economy.

Also related to our model are the theoretical contributions of Bruche and Llobet (2013), Hu and Varas (2021), and Begenau et al. (2021) investigating the role of bank incentives as driver of zombie lending, with a particular emphasis on the role of hidden losses and asymmetric information. We share with these papers the emphasis on developing a micro-founded model of bank lending, although we do not directly model asymmetric information frictions. Rather, we focus on developing a tractable general equilibrium framework to study how credit allocation and policy actions shape aggregate outcomes. In this regard, Tracey 2021 also shows that excessive forbearance may ultimately play a significant role in explaining the slow economic growth observed in the aftermath of aggregate shocks.

More broadly, our paper is related to the macroeconomic literature on financial frictions and misallocation (e.g., Midrigan and Xu 2014 and Buera et al. 2015). Gopinath et al. 2017, Banerjee and Hofmann 2018 and Asriyan et al. (2021) stress how a low interest rate environment and financial frictions can induce capital misallocation and aggregate losses. Our focus is on the central role played by financial intermediaries’ and how their actions depend on and influence macro policies. Buera et al. (2013) study how policies aimed at stimulating output can lead to long-run productivity losses. Their focus is on targeted industrial policies such as credit subsidies directly aimed at firms. In contrast, we are interested in studying stabilization policies and bank lending incentives in response to macroeconomic shocks. Importantly, while the key friction in their framework is the assumed inertia of policies (e.g., because of political capture), we show that in the presence of intertemporal congestion externalities, policymakers can fall into a policy trap even if they update policies optimally at each point in time. Finally, Li and Li (2021) and Crouzet and Tourre (2021) also highlight how policy interventions can have negative long-run effects, by worsening the quality distribution of firms or exacerbating debt overhang, respectively.
2 A Model of Zombie Lending

We begin with a static (single-period) model, which can be viewed as one period of the dynamic model presented in Section 4.

2.1 Environment: Heterogeneous Firms and Banks

The economy is populated by heterogeneous firms that differ in their productivity and risk. These firms’ investments require credit, which is provided by heterogeneous banks that differ in their level of capitalization. Figure 2 shows a timeline of the events in a period.

![Timeline of events within a period.](image)

2.1.1 Firms

There are two types of firms, $G$ or $B$. Initially, the economy is populated by a unit mass of incumbent firms. A mass $(1 - \lambda)$ of incumbents are endowed with an indivisible project of type $G$, that yields revenues $y^G(z)$ with probability $\theta^G$ and zero otherwise. A mass $\lambda$ of incumbents are endowed with type $B$ projects, yielding revenues $y^B(z)$ with probability $\theta^B$ and zero otherwise. $y^G$ and $y^B$ can depend on an aggregate shock $z$; we omit the dependence in $z$ until Section 3. There are also potential entrants, each endowed with a type $G$ project. Without loss of generality, we assume the mass of potential entrants to be equal to $\lambda$ to simplify expressions.

Both types of projects require $1$ in capital to be implemented. Firms have no wealth, and need to finance their project entirely via bank debt. Firm types are observable to banks. Therefore, the debt contracts feature type-specific interest rates: $G$ firms borrow at a rate $R^G$ and $B$ firms borrow at a rate $R^B$. 
In addition, all firms incur a production cost \((c + \epsilon)\), where \(c\) is common to all firms while \(\epsilon \in [0, \bar{\epsilon}]\) is an idiosyncratic cost shifter distributed according to the same c.d.f. \(H\) for both types of firms. The realization \(\epsilon_i\) is known to the firm (but not to the bank) before production and financing decisions are made. Potential entrants also draw an idiosyncratic cost shifter \(\epsilon\), observed before the decision of whether to enter or not, from the same distribution \(H\) as incumbents. Relative to incumbents, potential entrants must pay an additional entry cost \(\kappa \geq 0\) in order to be able to enter and produce, but they also have a technological advantage \(\gamma \geq 0\), that decreases their production cost to \((c - \gamma + \epsilon)\). For simplicity, \(\gamma\) is assumed to be equal to \(\kappa\), as in Caballero, Hoshi and Kashyap (2008).

Given the binary payoff structure, the project and the loan share the same risk: firms repay their loan entirely if their project succeeds, and default on the full loan if their project fails. We make the following assumption on payoffs:

**Assumption 1.** \(\Delta \theta = (\theta^g - \theta^b) > 0\) and \(0 \leq (\theta^b y^b - c) < (\theta^g y^g - c - \bar{\epsilon})\).

Type \(B\) projects are riskier, which captures the fact that \(B\) firms have more outstanding debt and are thus more likely to default on their new loans. Moreover, even the least productive type \(G\) firms with a draw \(\epsilon = \bar{\epsilon}\) are better than the most productive type \(B\) firms with a draw \(\epsilon = 0\). The greater risk and lower profitability of type \(B\) projects mirror the characteristics of “zombie firms.”

### 2.1.2 Banks

There is a unit mass of heterogenous financial intermediaries (hereafter, banks) with a balance sheet scale of $1. Banks are indexed by their exogenous equity \(e\), distributed in the interval \([e_{\min}, e_{\max}]\) according to the c.d.f. \(F\), with \(0 < e_{\min} \leq e_{\max} < 1\). In Section 5 we allow \(e\) to be chosen endogenously subject to equity issuance frictions.

Each bank can invest its entire $1 in a single asset, which can be either a risky corporate loan or a safe asset. Banks can lend to a type \(j = \{b, g\}\) firm at rate \(R^j\), earning an expected return equal to \(\theta^j R^j\). Credit markets are competitive: loan rates \(R^j\) are taken as given by both firms and banks, and determined in general equilibrium. Alternatively, banks can invest in “safe assets”. We interpret safe assets as a broad class of assets held in banks’ portfolios that are generally safer than corporate loans, such as mortgages, reserves, Treasuries, or asset-backed securities. Safe assets are supplied elastically and pay a risk-free return \(R^f\) set by

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6Empirical studies document that zombie firms are riskier borrowers, as they tend to have higher leverage, lower net worth, higher interest rate coverage ratio, and lower profitability ratios than healthy firms (Hoshi 2006; Acharya et al. 2019).
monetary policy. Investment in safe assets does not produce output; results are unchanged if investments in safe assets yields a positive output lower than what is produced by lending to firms.

One can interpret the assumption that banks invest in a single asset more broadly as capturing bank specialization. Loans can be reinterpreted as being portfolios of loans in the same sector for which individual banks have acquired information and competences.\(^7\)

On the liability side, a bank with capital \(e\) needs to raise \((1 - e)\) of debt in order to invest. In equilibrium, debt holders require an expected return equal to \(R^f\). The actual contractual rate paid to debt holders by each bank, \(\tilde{R}^j\), depends on the riskiness of banks’ asset choice \(j\) and on the degree of government guarantees indexed by a parameter \(p\) set by policy, as we describe next. Specifically, we assume that debt holders are able to recover their principal with probability \(p \in [0, 1]\) if the bank defaults.\(^8\) Thus the contractual rate \(\tilde{R}^j\) on the debt of a bank that invests in asset \(j \in \{g, b, f\}\) needs to satisfy

\[
R^f = \theta^j \tilde{R}^j + (1 - \theta^j) p.
\]

2.1.3 Relationship Lending

A major source of zombie lending stems from weak banks willing to “extend and pretend”, by rolling over loans at subsidized rates to legacy borrowers that should be declared non-performing.\(^9\) We incorporate this element by breaking the symmetry between old and new borrowers in a parsimonious way. Banks start the model matched with a legacy borrower. A random fraction \(\lambda\) of banks have an outstanding \(B\) borrower, and the remaining \((1 - \lambda)\) banks have an outstanding \(G\) borrower.

Assumption 2. If a bank switches from its legacy \(B\) borrower to a new borrower, its equity falls from \(e\) to \((e - \delta)\), for some switching cost \(\delta \geq 0\).

\(^7\)See, e.g., Berger et al. (2017) and Paravisini et al. (2020). The assumption of full specialization could be relaxed by allowing banks to hold a portfolio of projects with correlated risks a la Vasicek (1977) without affecting the key message of the model.

\(^8\)An alternative formulation would assume that the net interest \((\tilde{R}^j - 1)\) is also guaranteed with probability \(p\). Our formulation yields simpler expressions throughout and is consistent with the typical insurance scheme offered to depositors (e.g., by the FDIC in the U.S).

\(^9\)The empirical literature documents that the credit extended by under-capitalized banks to poorly performing firms is granted at rates lower than justified by the credit risk of these borrowers. The subsidized nature of these credit transactions is one of the quintessential features of zombie lending. For this reason, Caballero et al. (2008) and most of the following literature use subsidized bank credit as a criterion to empirically identify zombie firms, and finds that their borrowing rates are often as low as those charged to the safest borrowers (Acharya et al. 2019; Schivardi et al. 2021).
The presence of a positive switching cost will prolong some borrower-lender relationships. The switching cost $\delta$ captures first and foremost the loss provisions that banks must put aside when declaring loans as non-performing; but $\delta$ is also meant to include the screening effort that the bank must spend when creating a relationship with a new borrower.\footnote{The efficiency of the debt resolution system affects the cost of insolvencies and the magnitudes of loan loss provisions. Therefore, bankruptcy reforms may alleviate the incidence of zombie lending (Becker and Ivashina, 2022). However, the benefits of such reforms depend on the level of bank capitalization, which determines the strength of banks’ zombie-lending incentives (Kulkarni et al., 2021).}

Indeed, banks will never want to switch from a legacy $B$ borrower to a new $B$ borrower, so the only switches that could be observed in equilibrium are towards a new $G$ borrower. This presumes some costly information gathering to learn which borrowers are indeed good. Our results extend to a more general switching cost structure, with costs $\delta_{ij}$ depending on both the legacy match $i$ and the new match $j$.

The distinction between legacy and new borrowers requires us to model lending relationships. First, we need to determine which outstanding borrower-lender pairs are continued, and which of them are broken so that the bank can lend to a new borrower. Second, we must specify the loan rates offered to legacy borrowers, as those can differ from the rates offered to new borrowers due to the hold-up problem.

We model the renegotiation between banks and legacy borrowers as follows. At the beginning of a period, before the idiosyncratic cost shock $e$ of the borrower is realized, a bank and its legacy borrower choose whether to stay matched or not, and what loan rate $\tilde{R}^i$ the legacy borrower must pay to the bank if they do remain matched, as follows:

- **Privately efficient separations:** We assume that continuation and separation decisions are privately efficient from the borrower-lender pair’s perspective. The pair separates if and only if the joint surplus of remaining matched is lower than the joint surplus outside the relationship, in which case the bank lends to a new borrower and the borrower seeks to borrow from a new bank. Formally, denote

$$\Delta S^i(e) = S^i(e) - \bar{S}^i(e)$$

the difference between the joint surplus inside and outside the relationship, respectively, for a legacy borrower of type $i$ and a bank with capital $e$, given policies and equilibrium rates. The relationship is broken if and only if $\Delta S^i(e) < 0$.

- **Nash bargaining:** A continuing borrower-lender pair renegotiates the loan rate and splits the surplus according to generalized Nash bargaining, with $\omega \in [0, 1]$ denoting
the share of the surplus appropriated by the firm and \((1 - \omega)\) the share accruing to the bank. Therefore, conditional on the relationship remaining in place, that is \(\Delta S^i(e) \geq 0\), the legacy rate \(\bar{R}^i(e)\) for a borrower of type \(i\) matched to a bank with equity \(e\) is given by \(\bar{R}^i(e) = R^i - \omega \frac{\partial}{\partial \bar{R}} \Delta S^i(e)\). We assume \(\omega = 0\), which allows us to keep track of only two loan rates instead of three, as legacy and new \(B\) borrowers pay the same rate
\[
\bar{R}^b = R^b.
\]
A positive firm bargaining power \(\omega > 0\) would further decrease \(\bar{R}^b\).\(^{11}\)

### 2.1.4 Policy instruments: \(R^f\) and \(p\)

Policymakers affect banks decisions through the choice of \(R^f\) and \(p\). They directly control the level of the risk-free rate \(R^f\) through conventional monetary policy. They also set the parameter \(p\), which influences banks’ cost of capital through the debt pricing equation (1): a higher degree of insurance \(p\) encourages risky lending by decreasing the associated cost of funds. Equivalently, a low \(p\) means a strong market discipline: debt holders respond to bank risk-taking by requiring a higher rate.

There are several complementary interpretations of the policy variable \(p\). A natural one is to view \(p\) as capturing the degree of insurance offered to depositors above and beyond the level needed to prevent bank runs (that we normalize to \(p = 0\)). Another is to think of \(p\) as indexing the leniency of bank closure policy: higher \(p\) means more regulatory forbearance. More broadly, \(p\) can be thought of as an unconventional monetary policy tool, such as the quantitative easing (QE) implemented by central banks starting from the Great Recession and onward. The common thread of these policies that is relevant in our framework is the impact on banks’ cost of external financing: in all cases, a higher \(p\) undermines market discipline and incentivizes bank risk-taking.\(^{12}\) Acharya et al. (2019) provides evidence showing how unconventional monetary policy actions (the OMT program in Europe) helped lowering banks funding costs at the cost of reducing market discipline, with substantial effects

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\(^{11}\)Low loan rates are also a way to ensure repayment of the zombie loans when the default probability depends on loan rates, as in the literature on credit rationing (Stiglitz and Weiss, 1981). Faria E Castro et al. (2021) focus on this mechanism and find empirical support in U.S. data.

\(^{12}\)We focus on the case of an homogeneous forbearance policy \(p\) which is independent of banks’ portfolio choices. A risk-sensitive \(p\) would alleviate risk-shifting incentives, but institutional features may prevent regulators from tailoring \(p\) to bank-specific conditions. For instance, \(p\) could be constrained to be the same across banks in different European countries in spite of large difference in the quality of their balance sheets. Moreover, risk-sensitive forbearance policies may not be incentive-compatible in a general environment with private information about assets and/or moral hazard in monitoring, as shown by Chan, Greenbaum and Thakor (1992).
on banks’ asset composition.

The two variables $R_f$ and $p$ impact banks’ decisions—and therefore credit allocation—through two different channels. The first channel is a standard bank lending channel, that is, the choice between investing in safe assets versus lending to the productive sector. A lower $R_f$ stimulates lending to both types of firms by decreasing the return of investing in safe assets relative to loans. Government guarantees subsidize riskier investments, thus a higher $p$ also stimulates lending to both types of firms, by lowering the cost of funds.

The second channel is the zombie lending channel, operating through the choice between lending to different types of borrowers. A higher $p$ not only makes lending in general more appealing, but it also increases the profits from loans to $B$ firms relatively more. These loans are riskier, thus a given subsidy $p$ lowers the cost of funds by more when lending to $B$ firms, through the term $(1 - \theta^b) p$ in (1). As we will show, the incentives to lend to one type of firm or the other are bank-specific, as they depend on bank capitalization.

### 2.2 Equilibrium: Diabolical Sorting between Banks and Firms

Since there is a unit mass of banks and each bank lends to at most one firm, in equilibrium we must determine both the aggregate amount of lending (banks who do not lend invest in safe assets) and the composition of lending. As we explain below, the highest level of aggregate output is achieved when there is maximal creative destruction. That is, all the type $B$ incumbent firms exit, and are replaced by more productive type $G$ entrants. We model the entry and exit process building on CHK, with the additional layer of banks’ portfolio choices. Equilibrium loan interest rates are the variables that adjust to bring about, or hinder, creative destruction.

**Firms’ entry and exit decisions.** Given the realization of production costs $\epsilon$ and the borrowing rates offered by banks, incumbent firms decide whether to produce or exit, and potential entrants decide whether to enter or not. Incumbents remain in business and undertake their project if and only if they expect positive profits, which happens if and only if the idiosyncratic cost realization $\epsilon$ is lower than a type-specific threshold $\tilde{\epsilon}^i$, $i = g, b$. A type $i$ incumbent drawing $\epsilon$ produces if

$$\epsilon \leq \tilde{\epsilon}^g = \theta^i \left( y^i - R^i \right) - c$$

(2)
and exits otherwise. The masses of active firms of type $G$ and $B$ are respectively

\[
m^g = (1 - \lambda) H (\tilde{e}^g) + \lambda H (\tilde{e}^g) = H (\theta^g (y^g - R^g) - c),
\]

\[
m^b = \lambda H \left( \theta^b (y^b - R^b) - c \right).
\]

$m^g$ and $m^b$ are the aggregate loan demands from each type of firm. There is no intensive margin adjustment as projects are all of unit size, but higher loan rates decrease aggregate loan demand at the extensive margin.

**Banks’ portfolio choice.** A bank with equity $e$ chooses among the three investment options (safe assets, lending to type $G$ firms, lending to type $B$ firm) to maximize expected profits. Taking as given $p$, the loan rates $R^g$, $R^b$ and the risk-free rate $R^f$, the bank solves

\[
\max_{j \in \{g, b, f\}} \theta^j \left[ R^j - \tilde{R}^j (1 - e) \right]
\]

\[
s.t. \quad \tilde{R}^j = \frac{R^f - (1 - \theta^j) p}{\theta^j}.
\]

The following proposition, proved in Appendix C, characterizes the solution of banks’ problem as a function of their level of capitalization.

**Proposition 1** (Diabolical sorting). Define the following equity levels:

\[
e^* = 1 - \frac{(\theta^g R^g - \theta^b R^b)}{p \Delta \theta}, \quad e^{**} = 1 - \frac{(R^f - \theta^g R^g)}{p (1 - \theta^g)}
\]

and $z = \frac{R^f - p (1 - \theta^g)}{p \Delta \theta}$. Suppose that condition (A.2) in the Appendix holds. Then $e^* \leq e^{**}$ and banks matched with a legacy $B$ borrower invest as follows:

(i) Banks with equity $e < e^* + z \delta$ lend to a type $B$ borrower at rate $R^b$.

(ii) Banks with equity $e \in (e^* + z \delta, e^{**})$ lend to a type $G$ borrower at rate $R^g$.

(iii) Banks with equity $e > e^{**}$ do not lend and invest in safe assets at rate $R^f$.

Other banks follow the policies (i)-(iii) with a threshold $e^*$ instead of $e^* + z \delta$.

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13 The only difference if condition (A.2) does not hold is that $e^* > e^{**}$, implying that region (ii) does not exist, i.e., no bank lends to a $G$ firm. This leads to an even more extreme diabolical sorting: low equity banks lend to type $B$ borrowers, and high equity banks invest in safe assets.
In the special case of costless reallocation $\delta = 0$, $e^* + z\delta$ collapses to $e^*$ and all banks follow the same policies. Proposition 1 shows that the solution of banks’ problem features a diabolical sorting of poorly-capitalized banks with low productivity firms. It takes loan rates as given, but will continue to hold once rates are determined in general equilibrium. Our sorting result is in line with the evidence cited in introductory remarks that poorly-capitalized banks are more likely to extend credit to risky and unproductive borrowers.

There are two sources of sorting. The first is that regulatory forbearance induces risk-shifting in lending decisions, even when $\delta = 0$. Crucially, risk-shifting incentives depend on bank capitalization. To see this, we can rewrite bank $e$’s expected profits from choosing investment of type $j$ as

$$\theta^j[R^i - \tilde{R}^j(1 - e)] = \theta^jR^i - R^j(1 - e) + p(1 - \theta^j)(1 - e),$$

where the last term is the policy-induced subsidy to type $j$ investments. Government guarantees provide a subsidy only if banks have some leverage, and if they take positive risk; thus there is no subsidy for safe investments ($\theta^j = 1$) or for fully equity-funded banks ($e = 1$). Outside these extreme cases, the subsidy is increasing in $p$, in leverage $(1 - e)$ and in risk $(1 - \theta^j)$, but also supermodular in these three variables. Thus our diabolical sorting result reflects the strong complementarity between leverage and risk-taking: for any $p > 0$, banks with high leverage are the ones who benefit the most from risk-taking.

The second source of sorting is relationship lending: positive switching cost $\delta$ further increases zombie lending at the bottom of the bank equity distribution. Some banks with capitalization between $e^*$ and $(e^* + z\delta)$ choose to roll over the loan to their legacy $B$ borrower in order to economize the cost $\delta$, even though given their capital they would lend to a new $G$ borrower absent this preexisting lending relationship. The most interesting implications of this “evergreening” channel arise when we consider how it interacts with capital requirements in Section 5.

**General equilibrium.** Given policies $(R^f, p)$, the static general equilibrium of the model is characterized by loan rates $(R^a, R^b)$ such that agents optimize and credit markets clear. In the special case $\delta = 0$ the market clearing conditions are

$$F(e^*) = m^b = \lambda H \left( \theta^b \left( y^b - R^b \right) - c \right),$$  

$$F(e^{**}) - F(e^*) = m^a = H \left( \theta^g \left( y^g - R^g \right) - c \right),$$
where the thresholds $e^*$ and $e^{**}$ are defined in Proposition 1. The first line equalizes the supply of zombie loans, by banks with equity $e < e^*$, to the demand from type $B$ firms with idiosyncratic shocks $\epsilon \leq \theta^b (y^b - R^b) - c$. The second line does the same for good loans. We detail the market clearing conditions in the general case $\delta \geq 0$ in Appendix B.

### 2.3 Output and Policy Objective

Given equilibrium loan rates, aggregate output can be written as

$$Y = \int_0^{\theta^g (y^g - R^g) - c} [\theta^g y^g - c - \epsilon] dH(\epsilon) + \lambda \int_0^{\theta^b (y^b - R^b) - c} [\theta^b y^b - c - \epsilon] dH(\epsilon).$$

The first term in (6) captures the net contribution of type $G$ firms (both incumbents and entrants). The second term is the net contribution of type $B$ firms. By Assumption 1 both terms are positive, thus lower lending rates $R^g$ and $R^b$ increase aggregate output by stimulating the entry and continuation of productive firms.

We define potential output $Y^*$ as the highest possible aggregate output the economy can achieve given its fundamentals, given by $Y^* = \theta^g y^g - c - E[\epsilon]$. According to equation (6), the economy attains $Y^*$ when all bank capital is used to finance the productive sector (i.e., there is no investment in bonds) and, within the productive sector, the most productive firms (i.e., there is no zombie lending).

**Congestion Externalities and Policy Objective.** When analyzing optimal policy, we assume policymakers set policies $(p, R^f)$ to maximize output net of congestion externalities resulting from zombie lending. As we discussed in Section 1, a substantial body of empirical evidence highlights that zombie firms can impact the performance of healthier firms in the economy through various channels, such as congestion in labor and input markets (Caballero, Hoshi and Kashyap, 2008), congestion in output markets due to price competition (Acharya et al., 2020a), or reduced innovation incentives (Schmidt et al., 2020). Qualitative evidence of such distortions comes from industry-level data. For both Japan and peripheral European countries, Figure 3 shows a strong negative correlation between the change in the zombie share of an industry and its productivity growth, likely capturing a combination of negative externalities on healthy firms as well as lower creative destruction. Quantitatively, in Japan, CHK finds that, depending on the industry, the presence of zombies reduced other firms’ cumulative investment and employment by 14 to 50 pp. and 5 to 19 pp., respectively. In the aftermath of the European sovereign crisis, Acharya et al., 2019 and Blattner et al.
(2020) estimate that the reallocation of credit toward zombies can explain a 3 to 11 pp. employment loss of non-zombie firms experienced and a substantial portion of the observed decline of aggregate TFP.14

Figure 3: Zombies and productivity growth in Japan and Europe: Industry correlation

Note: On the y-axis is the growth rate (in percent) of TFP of a given industry. The x-axis reports the percentage change in industry share of zombie firms. Each dot is an industry (in the left panel) or an industry-country pair (in the right panel). The line is the best fit quadratic. For Japan, the change in industry share of zombie firms is the difference between average share of zombie firms in the 1981–1992 period and the average share of zombie firms in the 1993–2002 period; the TFP growth rate is the average annual growth rate between 1990 and 2000. For Italy, Spain, and Portugal the change in industry share of zombie firms is the difference between 2012 and 2015; the TFP growth rate is between 2012 and 2016. See Appendix A for information on the data sources.

As our focus is on banks’ incentives to engage in zombie lending and how policy affects them, we introduce these various congestion externalities in a reduced-form manner:

$$Y - \beta \Gamma(m^b),$$

where $\Gamma$ is an increasing and convex function of the extent of zombie lending $m^b$, and there is no externality if $m^b = 0$, that is, $\Gamma(0) = 0$; in Section 4 we revisit the role of externalities and provide a dynamic specification that more closely matches the empirical evidence. The parameter $\beta \geq 0$ denotes the policy weight on congestion externalities.

We assume that if different combinations of $p$ and $R^f$ can achieve the same level of output net of congestion externalities (7), policymakers strictly prefer policies that minimize $p$.15 With these considerations in mind, we define the optimal policy as follows:

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14Chari et al. (2021) and Cong et al. (2019) provide evidence for large emerging economies such as India and China.

15In general, welfare also depends on social costs associated with the end of some existing lending relationships and on fiscal costs of insuring banks at the expense of taxpayers. As mentioned above, we abstract
**Definition 1.** The optimal policy is the combination \((p, R^f)\) that minimizes \(p\) among the set of policies that maximize \(Y - \beta \Gamma (m^b)\).

### 3 Optimal Policy Response to Aggregate Shocks

Having set up the model and the policy objective, we now analyze how policymakers can optimally combine their instruments \(R^f\) and \(p\) to maximize their objective (7), and how the optimal policy mix should respond to shocks to fundamentals.

**Shocks.** We assume that an aggregate productivity or demand shock \(z\) hurts the revenue of good projects without affecting bad projects, whose revenues in case of a success remain at a fixed level \(y^b\). Therefore potential output \(Y^*\) also decreases with \(z\). We parametrize

\[
y^g(z) = \bar{y}^g (1 - z),
\]

where \(z\) lies between 0 and \(z_{\text{max}}\) such that Assumption 1 holds even for \(z = z_{\text{max}}\). More generally, all the results in this section apply to “gap-reducing” shocks \(z\) such that

\[
\theta^g \frac{dy^g}{dz} < \theta^b \frac{dy^b}{dz}.
\]

(8)

A central insight from our model is that such shocks—which bring the productivity of good firms closer to that of bad firms—are pernicious in terms of bank incentives they engender, and impose a significant constraint on policymakers trying to stimulate bank lending.

#### 3.1 Optimal Unconstrained Policy

A policy combination that achieves \(Y = Y^*\) must be optimal according to criterion (7) since output is then maximized and there are no congestion externalities either as \(m^b = 0\).

The next proposition shows that achieving potential output requires both \(R^f\) and \(p\) to be sufficiently low so as to maximize the bank lending channel while preventing the zombie lending channel. A low risk-free rate \(R^f\) discourages substitution towards safe assets; a low subsidy \(p\) curbs the risk-shifting incentives of poorly-capitalized banks.

**Proposition 2.** Given a shock size \(z\), there exist a threshold \(\bar{p} > 0\) and a function \(\bar{R}^f (p; z)\) increasing in \(p\) and decreasing in \(z\) such that output reaches its potential, \(Y = Y^*(z)\), if and from the former by assuming that the costs \(\delta\) are purely private. Regarding the latter, we assume that the shadow cost of public funds is infinitesimally small, and hence it does not enter the policy objective (7).
Figure 4: Optimal policy.

Note: The gray area indicates all policy combinations \((p, R_f)\) that achieve \(Y^*\) for a given \(z\). The black dot (on the y-axis) denotes the optimal policy mix.

only if

\[
R_f \leq \bar{R}_f(p; z) \quad \text{and} \quad p \leq \bar{p}.
\]

For any \(p > \bar{p}\), zombie lending necessarily emerges in equilibrium and output falls short of \(Y^*(z)\).

The proof in Appendix C includes closed-form expressions for the thresholds \(\bar{R}_f(p; z)\) and \(\bar{p}\). Figure 4 provides a graphical representation of the result in the \((p, R_f)\) space. The condition \(R_f \leq \bar{R}_f(p)\) ensures that the return on safe assets is sufficiently low to make lending attractive to the banks with the highest level of equity \(e_{\text{max}}\), as these are the banks who benefit the least from government guarantees. The second condition \(p \leq \bar{p}\) prevents the emergence of zombie lending, by ensuring that even the most leveraged banks, with equity \(e_{\text{min}}\), prefer to lend to type G firms.

An immediate consequence of Proposition 2 is that without any constraint on conventional monetary policy, a sufficiently low \(R_f\), together with \(p = 0\), achieves potential output at no insurance cost to the taxpayers and without any congestion externality, since the mass of zombie firms \(m^b\) remains at zero.\(^{16}\)

**Corollary 1.** Absent constraints on conventional monetary policy \(R_f\), potential output \(Y^*(z)\) can always be attained and the optimal policy sets \(R_f(z) = \bar{R}_f(0; z)\) and \(p(z) = 0\).

\(\bar{R}_f(0; z)\) provides a notion of the “natural interest rate”, that is the interest rate required to

\(^{16}\)A positive level of insurance may be desirable in order to prevent panic withdrawals, bank runs, and the costly liquidations of financial institutions that might follow. Our variable \(p\) should be interpreted as the insurance and forbearance that goes above and beyond the “normal” level of guarantees needed to ensure financial stability.
achieve $Y^*$ without subsidy. Crucially, the natural rate fluctuates with fundamentals: negative productivity or demand shocks in the form of declines in $y^d$ must be accommodated by a lower risk-free rate, exactly as in standard macroeconomic models. If shocks are large enough, the required natural rate $R^f$ may be excessively low.

3.2 Optimal Policy with an ELB Constraint

We now characterize the optimal joint policy response to shocks when $R^f$ cannot be made arbitrary low. We show that, in the presence of an effective lower bound that constrains conventional monetary policy, some degree of accommodation through a positive level of forbearance $p$ is desirable. Interestingly, we find that the optimal level of forbearance may be a non-monotonic function of the shock, if congestion externalities are sufficiently strong.

As discussed above, potential output can in principle be attained by conventional monetary policy alone by setting $R^f$ to a sufficiently low level. However, an “effective lower bound” (ELB) may prevent the central bank from implementing $Y^*$ if the required risk-free rate is too low. We now suppose there is an exogenous lower bound on the risk-free rate:

$$R^f \geq R^f_{\text{min}}. \tag{9}$$

The planning problem is to maximize welfare $Y - \Gamma(m^b)$ subject to the equilibrium conditions (4)-(6) and the ELB constraint (9). Our formal results focus on the tractable case of costless switching $\delta = 0$ and we discuss how our results extend to the general case $\delta > 0$.

We focus on two polar cases that are sufficient to illustrate the economic mechanisms: (i) when the weight $\beta$ on congestion externalities is low, in which case the optimal policy is just to maximize bank lending; and, (ii) when $\beta$ is high, in which case the optimal policy features “No Zombie lending”. Between these extremes, the optimal policy may be characterized by an interior solution that trades off the marginal congestion externality $\beta \Gamma'(m^b)$ with the output loss from reducing bank lending, and thus requires an intermediate level of accommodation.

Low $\beta$: Output-maximizing policy. If the welfare weight on congestion externalities is low enough relative to the output gains from zombie lending

$$\beta \Gamma'(1) < (\theta^by^b - c - \bar{c}) \tag{10}$$
then the optimal policy simply maximizes bank lending and output, by setting $p$ at a high enough level to ensure that no bank invests in safe assets.

There are two thresholds $z$ and $\bar{z}$. In line with Proposition 2, following small shocks $z \leq \bar{z}$, an accommodative conventional monetary policy can achieve $Y^*$ at no costs ($p = 0$). Moderate shocks $z \in [\bar{z}, \bar{z}]$, however, require combining conventional monetary policy and forbearance policy in order to keep the economy at its full capacity. Specifically, a positive $p$ helps stabilize output once conventional monetary policy is constrained by the lower bound ($R_f = R_{f\text{min}}$). In this region, the optimal unconventional policy is to expand regulatory forbearance in response to more severe shocks. The increase in $p$ subsidizes bank lending as much as possible, but all the lending is to type $G$ firms. Thus if shocks are moderate, some forbearance $p > 0$ is sufficient to attain $Y^*$.

Once the shock is severe enough, $z > \bar{z}$, stimulating aggregate lending necessarily triggers some zombie lending by banks at the bottom of the equity distribution.

**High $\beta$: No-Zombie Lending policy.** If congestion externalities are costly enough:

$$\beta \Gamma^*(0) > (\theta^b y^b - c), \quad (11)$$

then the marginal output gain from zombie lending is not worth bearing congestion externalities. The optimal policy is then to maximize output while preventing zombie lending, i.e., keeping $m^b = 0$. The striking result in this case is that the optimal forbearance policy $p(z)$ is non-monotonic in the size of the shock.

As in the previous case, for shocks $z < \bar{z}$, the economy can achieve its potential $Y = Y^*(z)$. If the economy is hit by severe aggregate shocks $z > \bar{z}$, conventional monetary policy is still constrained by the effective lower bound, but now the optimal unconventional policy needs to balance two opposite forces. On the one hand, an increase in regulatory forbearance (higher $p$) spurs lending at the expense of investment in safe assets. On the other hand, if forbearance $p$ is too high, poorly-capitalized banks engage in zombie lending, which creates congestion externalities.

As a result, for large enough shocks, policymakers must optimally reduce the degree of regulatory forbearance $p$ as shock size $z$ increases, and allow some banks to retrench from lending and invest in safe assets instead. Aggregate output $Y$ necessarily falls short of its potential $Y^*(z)$. Put differently, when severe aggregate shocks hit the economy, policy should allow healthy banks to start hoarding safe assets, rather than “pushing on a string”; more accommodation would only trigger more zombie lending by the poorly-capitalized
banks. Our result shows that there exists a “reversal” level of unconventional policy above which further accommodation becomes contractionary, a counterpart to the “reversal interest rate” for conventional monetary policy (Brunnermeier and Koby, 2019).

Proposition 3 formalizes these results. The proof, including the definitions of the thresholds $z$ and $\bar{z}$, is in Appendix C.

**Proposition 3** (Optimal policy with ELB). Suppose $\delta = 0$. There exist thresholds $z > 0$ and $\bar{z} > z$ such that the optimal policy response to an aggregate shock $z$ is the following:

(i) For small shocks $z \leq z$, conventional monetary policy alone achieves $Y^*$. The optimal policy features $p = 0$ and an interest rate $R_f (z)$ that decreases with the size of the shock.

(ii) For moderate shocks $z \in (z, \bar{z}]$, unconventional policy $p$ can achieve $Y^*$. The ELB binds, $R_f = R_f^\text{min}$, and the optimal $p(z)$ increases with the size of the shock.

(iii) For larger shocks $z > \bar{z}$, $Y^*$ is not attainable. The ELB binds and the optimal $p(z)$

(a) increases with the size of the shock in the case (10) of a low $\beta$;

(b) decreases with the size of the shock in the case (11) of a high $\beta$.

Figure 5 illustrates the result by contrasting the two policy regimes and the level of output achieved by the two policies in the short run. The only difference arises for $p$ in the case of large shocks $z > \bar{z}$. The high-$\beta$ policy backtracks and reduces forbearance $p$ as shocks grow larger, whereas the low-$\beta$ policy keeps accommodating more and more until it hits the upper bound $p = 1$.

Proposition 3 highlights the role of large shocks. A lack of profitable investment opportunities for good firms is not only detrimental per se, but it also makes zombie lending more attractive to banks. Thus zombie lending tends to emerge after large gap-reducing shocks that hit economies with a weak banking sector. By reducing the profitability gap between zombie and good firms, such shocks make good firms look temporarily closer to zombies. As a result zombie lending becomes more appealing, since the subsidy $p (1 - \theta^b) (1 - e)$ now accounts for a relatively larger share of the expected profit. While this is optimal in the case of a low $\beta$, it must be prevented in the case of a high $\beta$, by tightening policy in spite of large shocks $z$.

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17Note that our results abstract from the frictions in the bankruptcy system that may also follow such large shocks. A massive wave of bankruptcies may lead to court congestion, fire sales, and widespread financial stress, calling for a richer set of policies than those we consider, such as those analyzed in Gourinchas et al. (2020) and Greenwood, Iverson and Thesmar (2020).
Figure 5: Optimal policy as a function of shock $z$ under high-$\beta$ and low-$\beta$ policy regimes.

Note: The left panel illustrates the optimal joint policy response ($R_f^p$ and $p$) when $\beta$ is high (solid black line) and when $\beta$ is low (red dotted line) as a function of the size of the shock $z$. The right panel illustrates the aggregate output $Y$ under the policy regimes and potential output $Y^*$ (dashed line).

**Complementarity between bank capital and stabilization policy.** Our model also highlights that the capitalization of the banking system not only plays a crucial role in determining the allocation of credit—as illustrated in Proposition 1—but also mediates the effectiveness of policy interventions following real economic shocks. The threshold shock size $\bar{z}$ in Proposition 3 depends on the minimal level of equity $e_{\min}$:

**Corollary 2.** An improvement in the health of weak banks (higher $e_{\min}$) leads to a more resilient economy, in the sense that policy can achieve $Y^*$ in response to a larger range of shocks $z \in [0, \bar{z}]$.

This result links the potency of monetary policy to the level of capitalization of the banking system, and is consistent with the evidence in Acharya et al. (2020b).

To summarize, the single-period theoretical framework reproduces some key empirical findings relating the allocative efficiency of credit markets, optimal policy actions, and the capitalization of the banking system, recognizing that zombie lending has real spillover effects in the form of negative externalities imposed by unproductive firms on the other firms in the economy. The following section studies the dynamic implications of zombie lending in the presence of these externalities.

### 4 Dynamic Model: Policy Traps and Sclerosis

Zombie lending is far from being a temporary problem. As discussed in the introduction, a growing body of empirical evidence suggests that the adverse effects of credit misallocation
due to the proliferation of zombie lending practices might be persistent and even compound over time. The Japanese stagnation taking place since the 1990s is a textbook case of this phenomenon (Hoshi and Kashyap, 2015). Figure 6, panel A shows that the growth rate of nominal GDP and aggregate productivity slowed down dramatically relative to the pre-real estate crisis. A similar pattern emerges when comparing the growth rates of GDP and aggregate TFP of the Southern European countries in the aftermath of the sovereign debt crisis to the growth rates observed in Germany over the same period (Figure 6, panel B). To incorporate these features we turn to a dynamic version of our model that emphasizes how the interplay of accommodative policies and zombie lending can lead to persistent output losses and policy traps. Our main result shows that in response to even transitory shocks, the economy can get stuck in a state of permanent low productivity and output (which we call “sclerosis”) with policymakers forced to implement a combination of low interest rates and high forbearance (which we call a “policy trap”). We then discuss how the economy can exit such a trap through a large bank recapitalization or an improvement in the productivity of good firms.

4.1 Dynamic Environment

To analyze the short-run and long-run implications of zombie lending, we provide a dynamic foundation for the congestion externalities $\Gamma(m^b)$ imposed by the presence of zombie firms. Empirical studies highlight the gradual effects of zombie lending on both zombie firms and good firms. As in Section 3.2, we assume the economy is hit by an adverse aggregate shock $z$ at time $t = 0$, which affects the productivity of type $G$ firms: $y^G_0 = \bar{y}^G(1 - z_0)$. Like before, the shock is gap-reducing: the productivity of type $B$ firms is unaffected by the shock, hence the shock reduces the difference between the two types of firms.
The full cost of keeping zombie firms alive materializes over time. The presence of type $B$ firms hurts the productivity of healthy firms in the next period:

$$y_{t+1}^q = \bar{g}^q (1 - z_{t+1}) \quad t \geq 0$$

where the endogenous shock $z_{t+1}$ increases with the extent of zombie lending in the previ-
ous period
\[ z_{t+1} = a m^b_t + \eta^Z_{t+1}. \]  
\[ \eta^Z_{t+1} \] is an exogenous aggregate shock, and we will study the dynamics of the economy following an exogenous transitory shock \( z_0 = \eta^Z_0 > 0 \). In addition, productivity is now affected by an endogenous component \( a m^b_t \). The parameter \( \alpha \geq 0 \) is the counterpart of \( \Gamma' \) in the static model, capturing the strength of the congestion externalities.

**Bank and firm dynamics.** In the dynamic model, we need to specify how banks evolve over time. Bank returns are stochastic, with some banks failing and others making large profits. In general, accounting for bank entry and exit and tracking the evolution of the full distribution of bank equity presents significant technical challenges, similar to the ones encountered in macroeconomic models with heterogeneous households and incomplete markets. We thus make the following assumptions to make the dynamic model tractable:

**Assumption 3 (Bank dynamics).** There are overlapping generations of bankers: bank managers at \( t \) are replaced after one period and earn a fraction \( \rho \) of the income accruing at \( t + 1 \). The manager of a bank with date-\( t \) equity \( e_t \) chooses project \( i \in \{b, g, f\} \) to maximize
\[ \rho \theta^i \left[ R^i_t - \tilde{R}^i_t (1 - e_t) \right]. \]

At the beginning of each period \( t + 1 \), after date-\( t \) bank managers have been paid and replaced, failing banks are replaced by new banks and the profits of all surviving banks are pooled together and redistributed to all banks equally and banks raise equity \( \iota > 0 \).

This simplification allows us to keep track of the evolution of the aggregate capitalization of the banking system, rather than the entire distribution of bank equity. Since banks are indistinguishable, they will be indifferent between different investment options in equilibrium. Even though the portfolio of individual banks is indeterminate, the aggregate portfolio of the banking system is well-defined, which is all we need to study the output effects of zombie lending.

The short-term nature of bank managers’ contracts implies that banks’ franchise value does not enter the bank investment problem, therefore banks’ portfolio choice is the same as static problem of Section 2. In particular, given date-\( t \) equilibrium rates, the optimal portfolio choice is characterized by the same thresholds \( e^*_t \) and \( e^{**}_t \) stated in Proposition 1. In a more general setting, banks would have to consider their franchise value when choosing their portfolios, which would then feed back into the equilibrium thresholds \( e^*_t \)
and \(e_t^{**}\). Accounting for the effect of the franchise value on bank’s portfolio choices is an interesting extension that we leave for future research.\(^{18}\)

Like before, in each period, a fraction \(\lambda\) of incumbent \(G\) firms turn permanently into \(B\) firms, and there is a mass \(\lambda\) of type-\(G\) potential entrants.

**Equilibrium.** Given a path of policies \(\{R^f_t, p_t\}\) and fundamentals \(\{y^g_t, y^b_t\}\), a dynamic equilibrium is a sequence of masses \(\{m^b_t, m^g_t, m^f_t\}\), equity \(e_t\), and loan rates \(\{R^g_t, R^b_t\}\) such that for all \(t\), banks sort optimally, bank equity \(e_t\) follows Assumption 3, markets clear, and productivity follows (12).\(^{19}\)

Next, we describe how policies are determined depending on policymakers’ objectives, and characterize the resulting equilibria.

### 4.2 Policymakers’ Objective and Policy Rules

The dynamic equilibrium depends on the path of policies \(\{p_t, R^f_t\}\), which in turn are set by policymakers depending on their objective function. We assume that the policy objective is to maximize the present discounted value of aggregate output:

\[
\max_{\{p_t, R^f_t\}} \sum_t \beta^t Y_t.
\]

Unlike in Section 2, policymakers only care about output. The reduced-form externalities \(\Gamma(m^b_t)\) in the static model can be interpreted as the present value of future productivity losses for good firms due to current zombie lending. Lending to type \(B\) firms has short-term benefits but possible long-term costs, and the optimal policy depends on how much weight policymakers put on current lending relative to future productivity, as captured by \(\beta\). Therefore the welfare weight \(\beta\) put on congestion externalities in the static model can be micro-founded as the discount factor of policymakers.

As in the static model, we focus on two polar cases: a “No Zombie lending” policy under high \(\beta\), chosen by patient policymakers concerned about long-run productivity, and a short-termist or “myopic” policy under low \(\beta\), chosen by policymakers with strong reputational or electoral concerns, which makes them effectively more impatient. We interpret a

\(^{18}\)We also assume that firms are focused on short-term profits, hence their entry and exit decisions are the same as in the static model; unlike the assumption on the bank side which simplifies the dynamic model considerably, the assumption on firms is mostly for exposition and can be relaxed to allow for forward-looking firms, see Appendix D.

\(^{19}\)The full expressions defining optimal sorting, bank dynamics, and market clearing are in Appendix B.1.
low policy horizon as arising from term limits, regulatory capture by incumbents, or reputational concerns that create a wedge between the public and regulatory objectives, as analyzed, for example, by Boot and Thakor (1993). Another interpretation is to think of the short policy horizon as a reflection of policymakers’ inability to implement policies that have immediate fiscal costs. Fiscally constrained governments tend to help financial institutions in distress by deploying guarantees and/or engaging in some form of forbearance, rather than promptly intervening with capital injections or restructuring and resolution measures (Acharya et al., 2021a).

**Long horizon: No Zombie lending policy.** The No Zombie lending (NZ) policy $p^{NZ}(z_t, e_t)$ is the high-$\beta$ optimal policy in the static model described in Proposition 3, in the special case of a degenerate equity distribution with $e_{\text{min}} = e_{\text{max}} = e_t$. In particular, $p^{NZ}$ is non-monotonic in $z_t$. For moderate shocks (as long as $Y^*$ can be reached), regulatory forbearance $p$ increases with the shock $z_t$; for large shocks, the optimal $p$ decreases with $z_t$ (see Figure 5). Preventing zombie lending has a cost: it leads to a lower short-run output $Y_t$ than under the policy that maximizes short-run output (described next), as some healthy banks end up investing in safe assets instead of lending. A policymaker with a high enough discount factor $\beta$ is willing to bear this cost to maintain future productivity.

**Short horizon: Myopic policy.** Conversely, a policymaker with a sufficiently low discount factor $\beta$ chooses to minimize the short-term costs of the shock $z$. This might require allowing zombie lending in equilibrium, even if doing so jeopardizes future productivity and output. Specifically, the optimal myopic policy $p^m(z_t, e_t)$ is the low-$\beta$ optimal policy in Proposition 3. It maximizes short-run output at each point in time by ensuring that all banks lend ($m^q_t + m^b_t = 1$) but ignoring congestion externalities and future productivity losses. As a result, the optimal myopic $p$ is increasing in the size of the shock $z$: larger shocks are accommodated with a higher $p$, until $p$ reaches its upper bound of 1. Formally, we have $p^m(z_t, e_t) = \min \left\{ 1, \frac{P^m(z_t)}{1-e_t} \right\}$ where $P^m$ is an increasing function of $z_t$.

### 4.3 Persistence of Output Losses under Different Policy Regimes

We now turn to our main dynamic experiment and result: transitory shocks can generate permanent output losses and policy traps due to the dynamic externalities imposed by zombie lending. Suppose the economy starts in a “good” steady state in which the zero lower
bound is not binding: \( R = (\theta \bar{g} - c - \bar{\epsilon}) > R_{\text{min}}^f \). Thus no forbearance is needed \((\rho = 0)\), there is no zombie lending, aggregate output is \( Y = Y^* \), and equity is \( e_0 = \frac{\theta}{1 - (1 - \rho)\bar{g}} \).

At date-0 a transitory shock \( z_0 = \eta_0^Z > 0 \) hits, so that \( y_0^g = \bar{g}(1 - z_0) \). The shock only lasts for one period, hence we have \( \eta_t^Z = 0 \) for \( t \geq 1 \). We contrast the paths of the economy under the No Zombie lending and myopic policy rules. Recall from Proposition 3 that there exists a threshold \( \bar{z} \) such that for shocks \( z_0 \leq \bar{z} \), optimal policy can still attain the potential output \( Y^* \) without triggering any zombie lending. Therefore the NZ and myopic policies only differ for large shocks \( z_0 > \bar{z} \). Let us then restrict attention to large enough shocks \( z_0 > \bar{z} \). Under both policy stances, the optimal conventional policy implies setting the minimal risk-free rate \( R^f_t = R_{\text{min}}^f \) as long as \( z_t > \bar{z} \). However, the paths of \( p_t \) will differ across markedly. In fact, we show that seemingly small within-period differences between the NZ and myopic policies can lead to completely different long-run outcomes.

**No Zombie Lending Policy: Transitory Recession and Full Recovery.** Under the NZ policy (high \( \beta \)), congestion externalities never materialize since there is no zombie lending in any period in equilibrium. The endogenous component of productivity losses is always zero, and since there are no further exogenous shocks, \( z \) reverts immediately to zero starting from date-1 \((z_t = 0 \ \forall t \geq 1)\). The date-0 recession is “V-shaped”: it can be quite deep, but remains short-lived. Output recovers immediately from the transitory aggregate shock. The following proposition formally describes the full equilibrium path:

**Proposition 4.** Under the No Zombie lending policy, the transitional dynamics for policies and aggregate output following the shock \( z_0 \) are given by

\[
\begin{align*}
 t &= 0 \\
 R_0^f &= R_{\text{min}}^f \\
p_0 &= \frac{R_{\text{min}}^f + c - \theta \bar{y}^b}{(1 - e_0) (1 - \theta^b)} \\
Y_0 &= Y_0^{NZ} < Y^* (z_0)
\end{align*}
\]

\[
\begin{align*}
 \text{for all } t \geq 1 \\
 R_t^f &= \theta \bar{g} - c - \bar{\epsilon} \\
p_t &= 0 \\
Y_t &= Y^* (0)
\end{align*}
\]

**Myopic Policy: Policy Trap and Sclerosis.** Under a myopic policy regime (low \( \beta \)), policymakers accommodate using regulatory forbearance, and allow some zombie lending at any date \( t \), in spite of the potential long-term costs on the productivity of healthy firms.
The mass of zombies at date-\( t \) is

\[
m_t^b = \lambda H \left( \theta^b y^b - R_{ft}^f + p^m (z_t, e_t) \left( 1 - \theta^b \right) - c \right).
\]

In particular, since \( z_0 > \bar{z} \) the date-0 mass of zombies \( m_0^b \) will be positive, which hurts the productivity of good firms at date-1 through \( z_1 > 0 \), and so on. The form of congestion externalities (12) implies that \( z_t \) follows the first-order Markov process

\[
z_{t+1} = \alpha \lambda H \left( \theta^b y^b - R_{\text{opt}}^f (z_t) + p^m (z_t, e_t) \left( 1 - \theta^b \right) - c \right).
\]

The myopic policy creates an endogenous “reverse hysteresis” channel: current accommodation leads to endogenous persistence of the initial shock, that worsens when congestion externalities \( \alpha \) are larger. If \( \alpha \) is high enough, the myopic policy response to a sufficiently severe transitory shock \( z_0 \) pushes the economy to a steady state with permanently lower output, defined as follows:

**Definition 2** (Sclerosis steady state). A sclerosis steady state is a steady state equilibrium with the interest rate at the ELB \( (R^f = R^f_{\text{min}}) \), permanent forbearance \( (p > 0) \) and potential output permanently depressed \( (z > 0) \).

Unlike in standard macroeconomic models, the natural rate becomes an endogenous variable. Sclerosis is associated with a policy trap: present policies aimed at minimizing short-term losses tie the hands of future policymakers through their effect on future productivity. As a result, the economy may be stuck at the ELB forever even though the natural interest rate would recover to a positive level under a different policy rule.

We can now express our main dynamic result.

**Proposition 5** (Myopic policy and sclerosis). Suppose that congestion externalities are large enough, \( \alpha \geq \bar{\alpha} \), for some positive \( \bar{\alpha} \) (given in Appendix C) and the technical condition (A.3) in the Appendix holds. Let \( z^* (\alpha) \geq \bar{z} \) be the smallest positive solution to

\[
z = \alpha \lambda H \left( \theta^b y^b - R_{\text{min}}^f + P^m (z) \left( 1 - \theta^b \right) - c \right).
\]

1. \( z^* (\alpha) \) is increasing in \( \alpha \).

2. There exists a unique stable sclerosis steady state. It features maximal forbearance \( p = 1 \) and permanent output losses \( z_\infty > 0 \) such that

\[
z_\infty = \alpha \lambda H \left( \theta^b y^b - R_{\text{min}}^f + (1 - e_\infty) \left( 1 - \theta^b \right) - c \right).
\]
where \( e_{\infty} = \frac{\bar{s}}{1-(1-\rho)R_{\text{min}}} < e_0 \) denotes steady state bank equity.

3. For initial shocks \( z_0 < z^*(\alpha) \), the economy converges to the no-zombie steady state, while for initial shocks \( z_0 > z^*(\alpha) \) the economy converges to the stable sclerosis steady state with \( z_t > 0, p_t > 0 \) and a binding ELB \( R_{t}^f = R_{\text{min}}^f \) for all \( t \) along the transition.

Figure 7 displays the impulse responses of output losses \( z_t \), aggregate output \( Y_t \), and the optimal policies \( R_t^f \) and \( p_t \) under the two policy regimes (NZ policy, in black, and myopic policy, in red). Panel A shows equilibrium paths following a shock \( z_0 \) that is above \( \check{z} \) but below the threshold \( z^*(\alpha) \) defined in Proposition 2. The ELB binds at the time of the shock under both policy regimes. Forbearance also increases in both cases, but by much more under the myopic policy.

As a result, output drops sharply under the NZ policy, but recovers immediately to its pre-shock level at \( t = 1 \). The interest rate also recovers after the initial shock. By contrast, the myopic policy succeeds in stabilizing date-0 output at a higher level thanks to the more generous forbearance policy that keeps some zombie firms alive. The stabilization of short-term output comes at the cost of a protracted output loss for multiple periods, with interest rates stuck at the ELB and forbearance \( p \) at a high level. While this path features endogenous persistence of the initial shock, the economy eventually converges back to its pre-shock steady state.

Panel B shows the equilibrium paths following a large initial shock \( z_0 > z^*(\alpha) \). While initially the paths under the two policies regimes are similar to the ones following a smaller initial shock, they soon start diverging from each other. Like before, the economy experiences a sharp but short-lived output loss under the No Zombie lending regime. But under the myopic policy, the date-1 output loss \( z_1 \) stemming from congestion externalities is even larger than the initial shock \( z_0 \). This puts the economy on a dangerous path: at \( t = 1 \), the endogenously weaker fundamentals induce myopic policymakers to accommodate even further, by keeping interest rates as low as possible and allowing even higher forbearance \( (p_t^m > p_0^m) \), which, in turn, hurts date-2 productivity, and so on. For a while, this myopic policy manages to stabilize output \( Y_t \) close to potential output \( Y_t^* \), albeit with a major side effect: potential output \( Y_t^* \) itself (dashed red line) starts falling because the presence of zombie firms reduces the productivity other firms in the economy. Moreover, once zombie lending becomes a permanent feature of the economy, all policymakers can do is exert maximal accommodation to stimulate output \( (R_t^f = R_{\text{min}}^f, p^m = 1) \), which however is not sufficient to prevent a large gap between output and its potential. The economy snowballs towards sclerosis and monetary policy is trapped.
Figure 7: Impulse responses under the NZ policy (black) and the myopic policy (red dotted).

Panel A: Small initial shock $z_0 < z^*(\alpha)$

Output loss $z$

Output $Y$

Risk-free rate $R_f$

Forbearance $p$

Panel B: Large initial shock $z_0 > z^*(\alpha)$

Output loss $z$

Output $Y$

Risk-free rate $R_f$

Forbearance $p$

Note: Output losses $z_t$, aggregate output $Y_t$ and potential output $Y^*_t$ (dashed lines), and the optimal policies $R_f^t$ and $p_t$ under the two policy regimes (No Zombie lending, in solid black lines, and the myopic policy, in red dotted lines). Panel A: small initial shock $z_0 < z^*(\alpha)$. Panel B: large initial shock $z_0 > z^*(\alpha)$.
4.4 Exiting the Policy Trap

Proposition 5 characterizes the steady state for given fundamentals. Can an economy exit a policy trap and recover from sclerosis? An obvious way to exit the trap is to appoint a more conservative policymaker, as in the literature on inflation bias (Rogoff, 1985). In our context this would correspond, for instance, to switching from a myopic policy regime to a No Zombie lending policy regime. This is isomorphic to our earlier example; the only difference is that the initial shock $z_0$ is not exogenous but caused by the congestion externalities in the sclerosis steady state (that is, $z_0 = \alpha m^b_{\infty}$ where $m^b_{\infty}$ is the steady state mass of zombie firms). At date-0, the NZ policy reduces forbearance $p$ sufficiently to induce all zombie firms to exit. This causes a sharp but transitory recession, and allows a clean start at $t = 1$.

More interestingly, suppose we maintain the myopic policy regime but consider different initial conditions. We consider two experiments. In each case the economy starts from a policy trap with $R_f = R_{f,\text{min}}$ and $p = 1$, hence from the associated sclerosis steady state with output losses $z_{\infty}$.

**Bank recapitalization.** Suppose that at $t = 0$ the government recapitalizes the banking sector. In our model this corresponds to an exogenous increase of bank equity from $e_{\infty}$ to a higher level $e_0$. A small intervention will only have a transitory effect. But a large recapitalization can help the economy exit the policy trap. Figure 8, panel A, shows such an example. Output falls at the time of the recapitalization: $z_0$ is still high initially, hence lending opportunities are still weak and the higher equity induces a subset of banks to invest in safe assets. However, a better capitalized banking sector implies that risk-shifting incentives and zombie lending fall, which triggers a virtuous feedback loop: congestion externalities are lower in the next period, which makes lending to good firms more attractive, and so on. Over time, the economy can recover and converge back to the “good” steady state with no zombie lending, high interest rate, no forbearance, and high productivity ($z = 0$).

Historically, recapitalizations of the banking sector by the government—either directly through capital injection or indirectly at times through the establishment of “bad banks”—have been the most effective antidote to the proliferation of zombie lending.\textsuperscript{20} Despite their efficacy, decisive interventions have been more the exception than the norm. In both Japan and southern Europe, for example, despite policymakers’ recapitalization efforts the capita-

\textsuperscript{20}Noteworthy cases of successful recapitalization efforts through asset purchases by a “bad bank” are the establishment of the Korea Asset Management Corporation (KAMCO) in South Korea following the 1997–1998 financial crisis and the establishment of the Resolution Trust Corporation (RTC) in the U.S. following the Savings and Loans crisis in the 1980s.
Figure 8: Impulse responses under the myopic policy.

Panel A: Bank recapitalization at $t = 0$.

Output loss $z$

Output $Y$

Risk–free rate $R_f$

Forbearance $p$

Panel B: Improvement in $\bar{y}$ at $t = 0$.

Output loss $z$

Output $Y$

Risk–free rate $R_f$

Forbearance $p$

Note: Output losses $z_t$, aggregate output $Y_t$, and the optimal policies $R^f_t$ and $p_t$ under the myopic policy regime. Panel A: bank recapitalization at $t = 0$. Panel B: permanent increase in $\bar{y}$ at $t = 0$. 
talization of the banking system effectively shrunk or did not increase enough to cope with the aggregate shocks hitting the economy (Figure 9). Furthermore, as shown empirically by Peek and Rosengren (2005) and Giannetti and Simonov (2013) in Japan, and Acharya et al. (2019) in Europe, the timid recapitalization measures put in place were unable to prevent the spread of zombie lending, as they were unable to effectively recapitalize the weakest financial institutions.

**Figure 9: Bank capital in Japan and Europe after the aggregate shock**

Note: On the y-axis, the figure reports the difference (in percentage points) between the aggregate capital ratio of the banking sector in each country between year $t$ and the capital ratio of the banking sector in the benchmark for that country. The x-axis reports time (in years) since the aggregate shock. For Japan, the aggregate shock is the burst of the real estate crisis (time $t = 0$ is 1990), the benchmark is aggregate capital ratio of the Japanese banking sector itself in the pre-shock period (1986–1989), and the capital ratio is measured as core capital over total assets. For Italy, Portugal, and Spain, the aggregate shock is the burst of the European sovereign debt crisis (time $t = 0$ is 2010), the benchmark is aggregate capital ratio of the banking sector in Germany, and the capital ratio is measured as Tier 1 capital over risk-weighted assets. See Appendix A for information on the data sources.

Implicitly, our previous analysis assumes that the government lacks the ability or the willingness to recapitalize the banks sufficiently, at least in the short run. This can stem from fiscal constraints, or from the fact that just like the instruments we already considered, bank recapitalizations are subject to policy myopia. Hence a government may optimally delay injecting equity if the costs of doing so (e.g., political backlash, heightened sovereign credit risk) are borne immediately while the benefits only materialize over time. Studying the optimal mix of policies as a function of the government’s fiscal capacity is an important extension for future research. For instance, once we take into account both the costs of subsidies (through $p$) and the costs of recapitalization, how should the government balance the two policies? Another consideration is the resulting moral hazard: while it is ex-post optimal to target equity injections towards banks below the threshold $e^*$, the anticipation
of such interventions would undermine bank incentives ex ante.

**Improvement in fundamentals \( \theta \tilde{y} \).** Fundamentals such as the productivity of good firms affect the threshold \( z^* (\alpha) \) in Proposition 2. For instance, \( z^* \) is increasing in \( \theta \tilde{y} \) and decreasing in the churn parameter \( \lambda \). A low growth environment is thus particularly dangerous: not only is potential output already low, but the economy is also more fragile and output is more susceptible to fall below potential due to zombie lending. Conversely, an improvement in \( \theta \tilde{y} \) can help the economy exit the policy trap and sclerosis; but once the economy is in a trap it needs a large shift in fundamentals. Figure 8, panel B, shows an example with a sufficiently large increase in \( \tilde{y} \) and thus the gap between the two types of firms. Lending to good firms becomes relatively more attractive, which again sets the economy on a virtuous path towards a good steady state.

5 **Extension: Equity Issuance and Capital Requirements**

Our framework highlights how an undercapitalized banking sector constrains policymakers, thereby making the economy more fragile in response to fundamental shocks. In the baseline model, we made this point taking the distribution of bank equity as given; we now consider how the distribution of bank equity itself responds to monetary policy, forbearance, and regulation. How do the conclusions change when banks can choose their capital structure? And if capital is endogenous, can regulators solve the misallocation of credit by forcing banks to raise more capital?

5.1 **Endogenous Bank Capital**

We first extend the static environment described in Section 2 by allowing banks to issue equity. The main result in this section is that tightening conventional monetary policy can reduce zombie lending, through a bank equity issuance channel.

Suppose banks start with a pre-issuance equity level \( e \) before deciding jointly how much equity they want to issue \( (\Delta \geq 0) \) and in which asset to invest (type G loans, type B loans, or safe assets). We assume costless switching \( \delta = 0 \) to start with and consider the interaction
of issuance and \( \delta > 0 \) in the next section. Thus bank \( e \) solves:

\[
\max_{j \in \{g, b, f\}, \Delta} \theta^j \left( R^j - \tilde{R}^j (1 - e') \right) - \kappa (\Delta) \\
\text{s.t. } e' = e + \Delta
\]

where \( \kappa \) is an increasing, convex, differentiable equity issuance cost function. Conditional on choosing project \( j \), the optimal equity issuance is

\[
\Delta^j = (\kappa')^{-1} \left( \theta^j \tilde{R}^j \right)
\] (13)

Accounting for their optimal equity issuance decisions, banks sort themselves into projects \( j \). The optimal equity issuance policy does not depend directly on a bank’s pre-issuance equity \( e \) because the cost \( \kappa \) is additive. Yet, in equilibrium, the amount of issuance issued by different banks varies with \( e \). Intuitively, \( e \) determines banks’ asset choices, which in turn affect the optimal equity issuance. Hence risk-shifting acts as a double whammy: banks that start with a lower level of capitalization issue less equity, because they will be the ones lending to relatively riskier borrowers even after issuing more equity. By contrast, banks that start with high capital internalize that they will be the ones lending to safer borrower or even investing in safe assets, and thus have incentives to issue more equity.

As in the baseline model, there is a diabolical sorting: poorly-capitalized banks engage in risk shifting and zombie lending. But now the equity thresholds \( e^* \) and \( e^{**} \) depend on the equity issuance margin, which leads to the following result:

**Proposition 6.** Given loan rates \( R^g, R^b \), an increase in \( R^f \) decreases \( e^* \). An increase in \( p \) raises \( e^* \) more than without equity issuance.

A sufficient condition for these comparative statics to also hold in general equilibrium (taking into account the adjustment of loan rates \( R^g \) and \( R^b \)) is that all banks lend, i.e., \( e^{**} > e_{\text{max}} \).

Proposition 6 uncovers a new relationship between zombie lending and conventional monetary policy when bank equity is endogenous. As previously discussed, when banks cannot choose their leverage—or, equivalently, when equity issuance costs are infinitely high—the level of \( R^f \) has no bite on banks’ relative returns from lending to good versus bad types of firms. Once equity issuance costs are introduced, however, a reduction in the monetary policy rate \( R^f \) increases the threshold \( e^* \), thereby increasing zombie lending.

A higher interest rate increases the returns on all assets and therefore encourages banks to issue more equity to take advantage of these higher returns. Our reduced-form formulation in which equity is limited by an issuance cost function \( \kappa \) makes this point particularly...
stark and simple. More generally, higher interest rates will increase equity issuance if the required return on bank equity does not adjust fully with the risk-free rate, as is the case empirically, so that higher interest rates make the cost of equity relatively lower.21

5.2 Capital Requirements and Evergreening

A key policy question in the face of prevalent zombie lending is whether tightening capital requirements is a good remedy. In light of our sorting result, improving the distribution of bank capital appears to be a natural solution to tilt credit allocation towards safer and more productive lending. The counterargument is that tighter regulation may backfire, by generating incentives for banks to extend and pretend out of fear of having to recapitalize to satisfy the requirement. In this section we present a framework to think about these issues. Our main result is that if switching costs \( \delta \) are high enough, and capital requirements are already strict, then tightening regulation further can worsen zombie lending through the evergreening channel.

Can higher capital requirements prevent zombie lending? We build on our equity issuance extension and allow for switching costs \( \delta > 0 \) as described in Section 2.1.3. In addition, the regulator can impose a capital requirement: post-issuance equity \( e' \) must remain above a floor \( \hat{e} \). Throughout this section we keep other policies \( R_f \) and \( p \) fixed (for instance, because the economy has already fallen into a dynamic policy trap) to focus on the effect of capital requirements. It is convenient to define

\[
\sigma (e') = \theta^g \left[ R^g - \bar{R}^g (1 - e') \right] - \theta^b \left[ R^b - \bar{R}^b (1 - e') \right]
\]

which represents the payoff difference between lending to a \( G \) firm and a \( B \) firm (ignoring any equity issuance costs) for a bank with post-issuance equity \( e' \). We restrict attention to parameters such that if the regulator sets a capital requirement low enough that it does not bind even for the bank with the lowest capital \( e = e_{\min} \) then that bank prefers to lend to a

21An important implication of this result is that the endogenous response of banks’ capital structure imposes an additional constraint on monetary policy. Moderate interest rates are needed to prevent banks from investing in safe assets instead of lending, as in the baseline model with exogenous equity. But there is a new force: lowering interest rates “too much” makes zombie lending more likely, by deterring equity issuance. Hence achieving potential output \( Y^* \) requires, as in Proposition 2, to set \( p \) and \( R_f \) low enough, together with a novel restriction that the risk-free rate \( R_f \) cannot be set too low either. Proposition 8 in the Appendix formalizes this result.
type-B firm. Formally,

\[ \sigma(\hat{e}) < \kappa(\hat{e} - e_{\text{min}} + \delta) - \kappa(\hat{e} - e_{\text{min}}). \]

(14)

for all \( \hat{e} \leq \min\{e_{\text{min}} + \Delta^b, e_{\text{min}} + \Delta^g - \delta\} \). Condition (14) means that there is indeed some zombie lending absent capital requirements. This is the only interesting case to consider, as otherwise capital requirements would be irrelevant for credit allocation and aggregate output, and introducing them would only create a deadweight loss in terms of equity issuance costs.\(^{22}\)

In the absence of any switching costs (\( \delta = 0 \)), it is straightforward to deter zombie lending completely: the regulator can just impose a capital requirement \( \hat{e} \) that is sufficiently high, and more precisely, above the equity threshold \( e^* \) in an equilibrium without zombie lending. Intuitively, the case of low enough switching costs must be similar to when there are no switching costs at all. Indeed, we find that for low enough \( \delta \), there always exists a sufficiently tight capital requirement \( \hat{e}^{NZ} \) (where \( NZ \) stands for No Zombie lending) that suppresses zombie lending altogether (\( m^b = 0 \)). Does it mean that we can always solve the zombie lending problem using capital regulation? We find that the answer is no. Surprisingly, when the switching cost \( \delta \) is high enough, no capital requirement can deter zombie lending completely: some positive equilibrium zombie lending is inevitable. In fact, the stronger result is that increasing capital requirements beyond some level can even backfire, by further encouraging zombie lending:

**Proposition 7.** Let \( \hat{e}^{NZ} \) solve \( \sigma(\hat{e}^{NZ}) = \kappa(\hat{e}^{NZ} - e_{\text{min}} + \delta) - \kappa(\hat{e}^{NZ} - e_{\text{min}}) \).

- If \( \delta < \Delta^g - \Delta^b \), then any capital requirement above \( \hat{e}^{NZ} \) suppresses zombie lending.
- If \( \delta > \Delta^g - \Delta^b \), then zombie lending is minimized by setting the capital requirement

\[ \hat{e} = 1 - \frac{\theta^g R^g - \theta^b R^b}{p\Delta \theta} \]

and increasing capital requirements above that level strictly increases zombie lending. No capital requirement can suppress zombie lending.

The case of a high \( \delta \) captures the evergreening motive of zombie lending. The intuition is as follows. A bank compares two options: recognizing the loss at a cost \( \delta \), which allows

\(^{22}\)In a dynamic setting, capital requirements could matter for future credit allocation even if they do not bind in the present. This is one rationale behind precautionary cyclical capital requirements.
a fresh start with a new $G$ borrower, or rolling over the loan to the legacy $B$ borrower. The second option allows to economize the switching cost $\delta$, and becomes especially attractive with a high $\delta$. Switching to a new borrower brings an additional cost if the bank is already poorly-capitalized: its equity will drop to $e - \delta$, which forces the bank to undertake a costly recapitalization to satisfy the requirement $\hat{e}$. Thus there is a set of banks for which the cost of recapitalization acts as an additional motive to roll over the zombie loan, and the set of such banks expands as the capital requirement $\hat{e}$ increases.

Proposition 7 highlights a subtle link between capital requirements and zombie lending. In particular, both cases are likely to be relevant because the “switching cost” $\delta$ and the threshold $\bar{\delta}$ depend on the country and industry of the borrower, and the history of the lending relationship. For instance, $\delta$ will be higher when there is more asymmetric information between banks and potential new borrowers, and when zombie debt has been accumulating for a longer time (as this increases the losses that banks would eventually recognize). Just like in the dynamic model, the longer policymakers wait before tackling the zombie lending problem, the harder it becomes to solve it. The case of high switching costs $\delta$ is consistent with some of the empirical evidence on the unintended consequences of capital requirements, for instance following the increase in capital requirements by the European Banking Authority in 2011, as documented by Blattner, Farinha and Rebelo (2020). Relatedly, Chopra, Subramanian and Tantri (2020) show that other regulatory actions such as ex-post bank cleanups can also trigger zombie lending if they are not accompanied by ex-ante bank recapitalization.

6 Conclusion

In this paper we develop a theoretical framework with heterogeneous firms and banks to study the feedback loop between bank under-capitalization, credit misallocation due to zombie lending, accommodative monetary policy and regulatory forbearance, and adverse aggregate outcomes such as permanent losses in growth and productivity. Our model generates linkages that are consistent with several features of aggregate and banking sector data characterizing the “lost decade” of Japan following its real estate crisis, and more recently, the aftermath of the sovereign debt crisis in southern Europe. Viewing the world through the lens of our model, to avoid delayed recoveries and persistent output losses, policymakers should avoid excessively “pushing on a string” of forbearance towards banks precisely when economies are hit by large shocks.
Our results have salient policy implications and suggest several directions for further research. A focal point of our model is the interaction between monetary and banking policy with the fundamental of firms and bank in the economy, potentially converting transitory shocks into lost decades. This risk is receiving increasing attention in the aftermath of the recent pandemic. The banking sectors of many countries have been recapitalized to absorb stressed level of losses; nevertheless, the unprecedented amount of distress of the production sector combined with and the adoption of lenient regulatory stances toward financial intermediaries has inevitably raised the specter of long-term economic stagnation from a zombification of the economy. Further empirical work is needed to better inform policy makers coping with large shocks on how to optimally resolve the trade-off between short-term versus long-term losses.

Finally, our study also suggests how properly designed capital injections in the bank sector can effectively tackle the incentives problems at the root of zombie lending. However, it is assumed in our framework, as in the real world, that governments lack the willingness or the ability to recapitalize the banks sufficiently in a timely fashion. This can be due to policy myopia or to binding fiscal constraints. Studying the optimal mix of macro-financial policies as a function of the government’s fiscal capacity remains an open question and an important extension for future research.

References


Online Appendix

A Data Sources

Japan. The share of zombie firms is computed by weighting firms by their assets from Caballero et al. (2008). Real GDP and aggregate TFP is from the Penn World Tables. The capitalization of the banking system is defined as the total adjusted core capital over total assets of Japanese banks. It is an authors’ elaboration using data from Fukao (2003, 2007). Adjusted core capital is defined as Core capital + Unrealized capital gains and losses – Estimated under-reserves – Deferred Tax Assets. Core capital is Tier 1 capital. The computation of unrealized capital gains and losses follows Fukao (2003, 2007) and is equal to 0.6*(Market Value Shares – Book Value Shares). Estimated under-reserves is loss reserves minus estimated loan losses. The calculation of Deferred Tax Assets follows (2003, 2007). The industry data used to compute the correlation between the percentage change in share of zombie firms in a given industry (1981–1992 average to 1993–2002 average) and the TFP growth (average growth rate between 1990 and 2000) comes from Caballero et al. (2008).

Italy, Spain, Portugal, and Germany. The share of zombie firms is computed using ORBIS data. Real GDP and aggregate TFP is from the Penn World Tables. Bank capital is the Tier 1 ratio (Tier 1 capital over risk-weighted assets) from the ECB statistics warehouse. The industry data used to compute the correlation between the change in share of zombie firms (from 2012 to 2016) and TFP growth (from 2012 to 2016) comes from ORBIS.

B Static equilibrium conditions with $\delta > 0$

The market clearing condition for new $B$ loans

$$(1 - \lambda)F(e^*) + \lambda F(e^*) \left[ 1 - H \left( \theta^b \left( y^b - \bar{R}^b \right) - c \right) \right] = \lambda H \left( \theta^b \left( y^b - R^b \right) - c \right) \left[ 1 - F(e^* + z\delta) \right]$$

so indeed for $\delta = 0$ we have the simple form

$$F(e^*) = \lambda H \left( \theta^b \left( y^b - R^b \right) - c \right)$$

as in the main text.
The market clearing condition for new $G$ loans is
\[
H (\theta^g (y^g - R^g) - c) = (1 - \lambda) [F(e^{**}) - F(e^*)] + \lambda [F(e^{**}) - F(e^* + z\delta)] + \lambda \left[1 - H \left(\theta^b (y^b - R^b) - c\right)\right] [F(e^* + z\delta) - F(e^*)]
\]
which also specializes to the simple form in the main text
\[
H (\theta^g (y^g - R^g) - c) = F(e^{**}) - F(e^*)
\]
for $\delta = 0$.

**B.1 Dynamic Equilibrium**

Given a path of policies $\{R^f_t, p_t\}_{t \geq 0}$ and fundamentals $\{y^g_t, y^b_t\}_{t \geq 0}$, a dynamic equilibrium is a sequence of masses $\{m^b_t, m^g_t, m^f_t\}_{t \geq 0}$, equity $e_t$, and loan rates $\{R^g_t, R^b_t\}$ such that for all $t$, banks sort optimally:

\[
m^b_t > 0 \implies e_t \leq e_t^* = 1 - \frac{\theta^g R^g_t - \theta^b R^b_t}{p_t (\theta^g - \theta^b)},
\]
\[
m^g_t + m^b_t < 1 \implies e_t \geq e_t^{**} = 1 - \frac{R^f_t - \theta^g R^g_t}{p_t (1 - \theta^g)},
\]
bank equity $e_t$ follows
\[
e_t = \frac{1}{1 - (1 - \rho) \left[m^f_{t-1} R^f_{t-1} + m^g_{t-1} \theta^g [R^g_{t-1} - \tilde{R}^g_{t-1} (1 - e_{t-1})] + m^b_{t-1} \theta^b [R^b_{t-1} - \tilde{R}^b_{t-1} (1 - e_{t-1})]\right]},
\]
where $m^i_{t-1}$ is the mass of banks investing in asset class $i \in \{b, g, f\}$ at $t - 1$, markets clear
\[
F(e^*_t) = m^b_t = \left(m^b_{t-1} + \lambda m^g_{t-1}\right) H \left(\theta^b (y^b_t - R^b_t) - c_t\right),
\]
\[
F(e^{**}_t) - F(e^*_t) = m^g_t = \left[1 - \lambda m^g_{t-1} + \lambda\right] H \left(\theta^g (y^g_t - R^g_t) - c_t\right),
\]
\[
1 - F(e^{**}_t) = m^f_t = 1 - m^b_t - m^g_t,
\]
and productivity follows (12).
C Proofs

Proof of Proposition 1. There are two cases to consider:

Case 1. A bank prefers lending to a type $G$ borrower at rate $R^g$ instead of lending to a type $B$ borrower if:
\[
\theta^g \left( R^g - \tilde{R}^g (1 - e) \right) \geq \theta^b \left( R^b - \tilde{R}^b (1 - e) \right).
\]

Using the definition of $\tilde{R}^j$, $j = g, b$, this condition is met for banks with level of capitalization above the following threshold:
\[
e \geq e^* = 1 - \frac{\theta^g R^g - \theta^b R^b}{p \Delta \theta}.
\]

When $\delta > 0$ and a bank has a legacy $B$ borrower, the bank prefers to switch to a new $G$ borrower if
\[
\theta^g \left( R^g - \tilde{R}^g (1 - e + \delta) \right) \geq \theta^b \left( R^b - \tilde{R}^b (1 - e) \right)
\]
which is equivalent to
\[
e \geq e^* + z\delta
\]
where $z = \frac{g^g \tilde{R}^g}{p \Delta \theta} = \frac{R^f - p(1 - \theta^g)}{p \Delta \theta}$.

Case 2. A bank prefers investing its capital in safe assets rather than lending to a type $G$ borrower at rate $R^g$ if:
\[
R^f - R^d (1 - e) > \theta^g \left( R^g - \tilde{R}^g (1 - e) \right)
\]

Using the definition of $\tilde{R}^g = \frac{R^d - p(1 - \theta^g)}{\theta^g}$ and under the assumption that $R^d = R^f$, this condition is met for banks with level of capitalization above the following threshold:
\[
e > e^{**} = 1 - \frac{R^f - \theta^g R^g}{p (1 - \theta^g)}.
\]

As long as $e^{**} > e^*$, a bank that prefers investing in safe assets over lending to type $G$ firms a fortiori prefers investing in safe assets over lending to type $B$ firms. The following conditions ensured that $e^* < e^{**}$:
\[
\frac{R^f - \theta^g R^g}{1 - \theta^g} \leq \frac{\theta^g R^g - \theta^b R^b}{\theta^g - \theta^b},
\]
or, equivalently,
\[ R^f \Delta \theta \leq \theta^g R^g (1 - \theta^b) - \theta^b R^b (1 - \theta^g). \]  

(A.2)

**Proof of Proposition 2.** \( Y = Y^* \) is achieved when all banks lend and there is no zombie lending, hence \( m^g = 1 \) and \( m^b = 0 \). Relative to this composition of firms, any substitution towards bonds decreases output, and any increase in zombie lending decreases output by Assumption 1.

In an equilibrium with \( Y = Y^* \) loan rates are given by

\[
\begin{align*}
R^b &= y^b - \frac{c}{\partial^b} \\
R^g &= y^g - \frac{1}{\partial^g} (c + \bar{\varepsilon})
\end{align*}
\]

Given these equilibrium loan rates, we verify that that all banks lend, that is \( e^{**} \geq e_{\text{max}} \), and that there is indeed no zombie lending, that is \( e^* \leq e_{\text{min}} \).

These conditions can be rewritten respectively as

\[
1 - \frac{R^f - \theta^g R^g}{\theta^g (1 - \theta^g)} = 1 - \frac{R^f + c + \bar{\varepsilon} - \theta^g y^g}{\theta^g (1 - \theta^g)} \leq e_{\text{max}} \iff R^f \leq \tilde{R}^f(p)
\]

where \( \tilde{R}^f(p) = \theta^g y^g - c - \bar{\varepsilon} + (1 - e_{\text{max}})(1 - \theta^g) p \), and when \( \delta = 0 \)

\[
1 - \frac{\theta^g R^g - \theta^b R^b}{\theta^g (1 - \theta^g)} = 1 - \frac{\theta^g y^g - \theta^b y^b - \bar{\varepsilon}}{\theta^g (1 - \theta^g)} \leq e_{\text{min}} \iff p \leq \bar{\rho}
\]

where \( \bar{\rho} = \frac{\theta^g y^g - \theta^b y^b - \bar{\varepsilon}}{(1 - e_{\text{min}})(\theta^g - \theta^b)}. \) When \( \delta > 0 \) the first condition \( R^f \leq \tilde{R}^f(p) \) is unchanged (recall that we assumed ending a relationship with a good firm is costless, and \( \delta \) only applies in the case of a legacy \( B \) borrower). The second condition \( p \leq \bar{\rho} \) ensuring no zombie lending becomes instead

\[
1 - \frac{\theta^g y^g - \theta^b y^b - \bar{\varepsilon}}{p \Delta \theta} + \frac{R^f - p(1 - \theta^g)}{\Delta \theta} \delta \leq e_{\text{min}}
\]

hence the only difference is that \( p \) must be below \( \bar{\rho} \) where

\[
\bar{\rho} = [(1 - e_{\text{min}})\Delta \theta + R^f \delta] \frac{1 - \sqrt{1 - \frac{4\delta(1 - \theta^g)(\theta^g y^g - \theta^b y^b - \bar{\varepsilon})}{(1 - e_{\text{min}})\Delta \theta + R^f \delta^2}}}{2\delta(1 - \theta^g)}
\]

which by l’Hôpital simplifies to the previous expression \( \frac{\theta^g y^g - \theta^b y^b - \bar{\varepsilon}}{(1 - e_{\text{min}})(\theta^g - \theta^b)} \) as \( \delta \to 0 \).

Moreover, if \( R^f \) is lower than the type \( G \) project with the lowest net present value, i.e.
\[ R^f < \theta^g y^g - c - \bar{\epsilon}, \] then all banks lend and with \( p \leq \bar{p} \) the economy reaches \( Y^* \) because there is also no zombie lending. Finally, if \( p > \bar{p} \) then there is necessarily zombie lending in equilibrium and \( Y < Y^* \), regardless of the level of \( R^f \).

**Proof of Proposition 3.** When the shock \( z \) is small, an accommodating conventional monetary policy alone can achieve \( Y = Y^* \) at no costs (\( p = 0 \)), without violating the ELB constraint. Adapting the results of Proposition 2, the monetary policy rate that achieves \( m^b = 1 \) with \( p = 0 \) is

\[ R^f (z) = \theta^g \bar{y}^g (1 - z) - c - \bar{\epsilon}. \]

This interest rate satisfies the ELB constraint if

\[ z \leq \bar{z} = 1 - \frac{R^f_{\min} + c + \bar{\epsilon}}{\theta^g \bar{y}^g} \]

For moderate shocks, \( z_t > z \), a combination of conventional and a lax forbearance policy, \( p(z) \), can still achieve \( Y = Y^* \) even if the ELB binds. Adapting the results of Proposition 2, given the loan rates in an equilibrium without zombie lending, this requires

\[ R^f (z) = \theta^g \bar{y}^g (1 - z) - c - \bar{\epsilon} + (1 - e_{max}) (1 - \theta^g) p(z). \]

Exhausting the stimulus from conventional monetary policy, the optimal policy sets \( R^f (z) = R^f_{\min} \), so \( p \) must satisfy \( R^f_{\min} = \theta^g \bar{y}^g (1 - z) - c - \bar{\epsilon} + (1 - e_{max}) (1 - \theta^g) p(z) \), or

\[ p(z) = \frac{R^f_{\min} + c + \bar{\epsilon} - \theta^g \bar{y}^g (1 - z)}{(1 - e_{max}) (1 - \theta^g)} \]

which is an increasing function of \( z \). The conjectured equilibrium loan rates are correct as long as

\[ z < \tilde{z} = 1 - \frac{1}{\theta^g \bar{y}^g} \left[ (\theta^b y^b + \bar{\epsilon}) (1 - e_{max}) (1 - \theta^g) + \left( R^f_{\min} + c + \bar{\epsilon} \right) (1 - e_{min}) (\theta^g - \theta^b) \right] \left[ (1 - e_{max}) (1 - \theta^g) + (1 - e_{min}) (\theta^g - \theta^b) \right]. \]

For large shocks, \( z > \tilde{z} \), conventional monetary policy is constrained by the lower bound and increasing the level of forbearance induces credit misallocation. There are two cases, depending on the value of the weight \( \beta \) on congestion externalities in social welfare (7):

- If \( \beta \) is high as in (11), the optimal policy response ensures \( m^b = 0 \) but \( Y^* \) is not
attainable. The optimal forbearance policy $p(z) > 0$ solves:

$$F \left( 1 - \frac{(1 - e_{\text{min}}) (\theta^g - \theta^b) + R^f_{\text{min}} + c - \theta^b y^b}{1 - \theta^g} \right) = H \left( \theta^g y^g (1 - z) - \theta^b y^b - p (1 - e_{\text{min}}) \Delta \theta \right),$$

which implies that the optimal $p(z)$ is decreasing in the size of the shock.

- If $\beta$ is low as in (10), the optimal policy response maximizes bank lending and thus output, ignoring congestion externalities. Given $z$ and $p$ the equilibrium loan rates $R^b(z, p)$ and $R^g(z, p)$ must solve

$$\lambda H \left( \theta^b \left( y^b - R^b \right) - c \right) = F \left( 1 - \frac{\theta^g R^g - \theta^b R^b}{p \Delta \theta} \right)$$

$$\lambda H \left( \theta^b \left( y^b - R^b \right) - c \right) + H \left( \theta^g \left( y^g(z) - R^g \right) - c \right) = F \left( 1 - \frac{R^f_{\text{min}} - \theta^g R^g}{p (1 - \theta^g)} \right)$$

If $e^{**}$ evaluated when $p = 1$ is strictly below $e_{\text{max}}$, i.e., $1 - \frac{R^f_{\text{min}} - \theta^g R^g(z, 1)}{1 - \theta^g} < e_{\text{max}}$, then the optimal $p$ is the maximal possible forbearance $p = 1$. Otherwise the optimal $p$ is the lowest $p$ ensuring that all banks lend, solving

$$1 - \frac{R^f_{\text{min}} - \theta^g R^g(z, p)}{p (1 - \theta^g)} = e_{\text{max}}$$

which yields a solution $p(z)$ that is increasing in $z$. To see this, we rewrite the equilibrium system under the optimal policy as

$$D_b(R^b) = S_b(R^b, R^g, p)$$

$$D_b(R^b) + D_g(R^g, z) = 1$$

$$(1 - e_{\text{max}})(1 - \theta^g)p + \theta^g R^g = R^f_{\text{min}}$$

where we define the demand for type B loans $D_b = \lambda H \left( \theta^b \left( y^b - R^b \right) - c \right)$, the demand for type G loans $D_g = H \left( \theta^g \left( y^g(z) - R^g \right) - c \right)$, and the supply of type B loans $S_b = F \left( 1 - \frac{\theta^g R^g - \theta^b y^b}{p \Delta \theta} \right)$. The argument only relies on the monotonicity of these functions and goes through even without differentiability, but the exposition is simpler using derivatives and the implicit function theorem. Suppose that the optimal forbearance is not everywhere non-decreasing with $z$, i.e., there exists a $z$ such that $\frac{dp}{dz} < 0$. Then
from the third line of the system we have \( \frac{dR^b}{dz} > 0 \). The second line then implies 
\[
\frac{dR^b}{dz} = -\frac{1}{\partial D_b} \left( \frac{\partial D_b}{\partial R^g} \frac{dR^g}{dz} + \frac{\partial D_g}{\partial z} \right) \leq 0.
\]

But this is incompatible with the first line
\[
\frac{dR^b}{dz} = \frac{1}{\partial R^g - \partial S_b^b} \left( \frac{\partial S_b}{\partial R^g} \frac{dR^g}{dz} + \frac{\partial S_b}{\partial p} \frac{dp}{dz} \right) \leq 0.
\]

Therefore \( p'(z) \geq 0 \) everywhere.

**Proof of Corollary 2.** We have from the expression of \( \hat{z} \) in the proof of Proposition 3
\[
\text{sign} \left( \frac{\partial \hat{z}}{\partial e_{\min}} \right) = \text{sign} \left( R^f_{\min} + c - \theta^b y^b \right)
\]
and \( R^f_{\min} + c - \theta^b y^b \geq 0 \) follows from the binding ELB constraint.

**Proof of Proposition 5.** We assume a technical condition on the distribution \( H \) of idiosyncratic cost shocks \( \epsilon \),
\[
\sup_{\epsilon \in [0,1]} h \left( \frac{\mathcal{H} \left( \theta^b y^b - R^f_{\min} + (1 - \epsilon) (1 - \theta^b) - c \right)}{\mathcal{H}^{-1} \left( 1 - \lambda H \left( \theta^b y^b - R^f_{\min} + (1 - \epsilon) (1 - \theta^b) - c \right) \right)} \right) \geq 1 - \frac{\Delta \theta}{1 - \theta^b} (A.3)
\]
which is satisfied when \( H \) is uniform, for instance.

A stable sclerosis steady state must have
\[
p^m(z, \epsilon_\infty) = 1
\]
i.e.
\[
H \left( \theta^b y^b (1 - z_0) - R^f_{\min} + (1 - e_0) (1 - \theta^b) - c \right) + \lambda H \left( \theta^b y^b - R^f_{\min} + (1 - e_0) (1 - \theta^b) - c \right) < 1
\]
This can be written concisely as
\[ z > Z(e_\infty) \]

where
\[ \zeta(e) = 1 - \frac{R^f_{\min} + c - (1 - e)(1 - \theta^g) + H^{-1} \left( 1 - \lambda H \left( \theta^b y^b - R^f_{\min} + (1 - e)(1 - \theta^b) - c \right) \right)}{\theta^g \bar{g}^g} \]

\[ Z(e) = \max \{ \bar{z}, \zeta(e) \} \]

are decreasing functions of \( e \) by (A.3).

At any \( t \) the zero lower bound binds and \( p^m(z_t, e_t) > 0 \) if and only if \( z_t \geq \bar{z} \). Moreover, if \( z_t \geq Z(e_t) \) then the optimal myopic policy sets \( p^m(z_t, e_t) = 1 \) and therefore
\[ z_{t+1} = z \left( \lambda H \left( \theta^b y^b - R^f_{\min} + (1 - e_t)(1 - \theta^b) - c \right) \right) \]

Thus we have a permanent sclerosis equilibrium (defined below) if for each \( t \)
\[ z_{t+1} \geq Z(e_{t+1}) \]

or
\[ z \left( \lambda H \left( \theta^b y^b - R^f_{\min} + (1 - e_t)(1 - \theta^b) - c \right) \right) \geq \max \{ \bar{z}, \zeta \left( t + (1 - \rho) R^f_{\min} e_t \right) \} \]

that is for all \( t \)
\[ \alpha \geq \frac{\max \{ \bar{z}, \zeta \left( t + (1 - \rho) R^f_{\min} e_t \right) \}}{\lambda H \left( \theta^b y^b - R^f_{\min} + (1 - e_t)(1 - \theta^b) - c \right)} \]

\( \zeta \) is decreasing in \( e \) but the denominator is also decreasing in \( e_t \). We always have
\[ \frac{t}{1 - (1 - \rho) R^f_{\min}} = \bar{e}_\infty \leq e_t \leq e_0 = \frac{t}{1 - (1 - \rho) [\theta^g \bar{g}^g - c - \bar{e}]} \]

Therefore an upper bound on the right-hand side is
\[ \hat{\alpha} = \frac{\max \{ \bar{z}, \zeta \left( t + (1 - \rho) R^f_{\min} e_\infty \right) \}}{\lambda H \left( \theta^b y^b - R^f_{\min} + (1 - e_0)(1 - \theta^b) - c \right)} \]

and a sufficient condition for permanent sclerosis to happen is \( \alpha \geq \hat{\alpha} \).
Proof of Proposition 6. Following the same steps as without equity issuance costs we find:

\[ e^* = 1 - \frac{\theta^g R^g - \theta^b R^b}{(\theta^g - \theta^b)} \frac{\varphi \left( \theta^g R^g \right) - \varphi \left( \theta^b R^b \right)}{\theta^g R^g - \theta^b R^b} \]

\[ e^{**} = 1 - \frac{R^f - \theta^g R^g}{p (1 - \theta^g)} \frac{\varphi \left( R^f \right) - \varphi \left( \theta^g R^g \right)}{R^f - \theta^g R^g} \]

where \( \varphi (x) = x (\kappa')^{-1} (x) - \kappa ((\kappa')^{-1} (x)) \). The function \( \varphi \) inherits the properties of \( \kappa \), as \( \varphi' (x) = (\kappa')^{-1} (x) \) and \( \varphi'' (x) = \frac{1}{\kappa''((\kappa')^{-1} (x))} \). Since \( \theta^g R^g - \theta^b R^b = (\theta^g - \theta^b) p > 0 \), it follows from the convexity of \( \varphi \) that the slope of \( \frac{\varphi \left( \theta^g R^g \right) - \varphi \left( \theta^b R^b \right)}{\theta^g R^g - \theta^b R^b} \) is increasing with \( R^f \) and (decreasing with \( p \)).

Proof of Propositions 7. When \( \delta > \Delta^g - \Delta^b \), there are three relevant regions for banks initially matched with a bad firm. If \( e < \hat{e} - \Delta^b \), then the capital requirement is binding even if the bank remains with its legacy \( B \) borrower. If \( e > \hat{e} - \Delta^g + \delta \), the capital requirement is never binding, whether the bank switches or not. For intermediate equity \( e \in [\hat{e} - \Delta^b, \hat{e} - \Delta^g + \delta] \), the capital requirement is binding only if the bank switches.

We start with the banks matched to a borrower that turns \( B \).

1. Suppose that \( \hat{e} \) is high enough that the bank \( e = \hat{e} - \Delta^b \) prefers to switch to a new \( G \) borrower and thus issue \( \hat{e} - e - \delta = \Delta^b + \delta \), that is

\[ \sigma (\hat{e}) \geq \kappa \left( \Delta^b + \delta \right) - \kappa \left( \Delta^b \right) \]  

or

\[ \delta \leq \kappa^{-1} \left( \sigma (\hat{e}) + \kappa \left( \Delta^b \right) \right) - \Delta^b \]

Therefore, all the banks above \( e = \hat{e} - \Delta^b \) will prefer to switch, and the only potential for zombie lending is for banks below \( \hat{e} - \Delta^b \). In that case, banks lending to zombies are those with pre-issuance equity \( e \) below the indifference threshold \( e^* \) solving

\[ \sigma (\hat{e}) = \kappa (\hat{e} - e^* + \delta) - \kappa (\hat{e} - e^*) \]

Note that \( \sigma (\hat{e}) > 0 \) implies \( \hat{e} > E^* \). From the implicit function theorem, when \( \hat{e} \)
increases (holding loan rates fixed in this partial equilibrium first step) we have
\[
\frac{\partial e^*}{\partial \hat{e}} = 1 - \frac{\sigma'(\hat{e})}{\kappa'(\hat{e} - e^* + \delta) - \kappa'(\hat{e} - e^*)} = 1 - \frac{\theta^g \tilde{R}^g - \theta^b \tilde{R}^b}{\kappa'(\hat{e} - e^* + \delta) - \kappa'(\hat{e} - e^*)}
\]
This can be rewritten as
\[
\frac{\partial e^*}{\partial \hat{e}} = 1 - \frac{\kappa'(\Delta^g) - \kappa'(\Delta^b)}{\kappa'(\hat{e} - e^* + \delta) - \kappa'(\hat{e} - e^*)}
\]
Since \(\delta > \Delta^g - \Delta^b\) and \(\hat{e} - e^* \geq \Delta^b\), we necessarily have
\[
\frac{\partial e^*}{\partial \hat{e}} > 0
\]
and thus in this region, increasing capital requirements worsens legacy zombie lending.

(a) Suppose then that (A.4) doesn’t hold:
\[
\delta > \kappa^{-1}\left(\sigma(\hat{e}) + \kappa\left(\Delta^b\right)\right) - \Delta^b
\]
which implies that the bank with \(e = \hat{e} - \Delta^b\) prefers to stay matched with its legacy B borrower.

i. If the bank with \(e = \hat{e} - \Delta^g + \delta\) prefers to switch to a new G borrower, that is
\[
\sigma(\hat{e}) > \kappa(\Delta^g) - \kappa\left(\Delta^b\right) + \theta^b \tilde{R}^b \left(\delta - \Delta^g + \Delta^b\right)
\]
holds, then all banks with even higher \(e\) also switch. Thus the indifference threshold \(e^*\) is in the intermediate region \([\hat{e} - \Delta^b, \hat{e} - \Delta^g + \delta]\) and solves
\[
\theta^b \left[R^b - \tilde{R}^b \left(1 - e^* - \Delta^b\right)\right] - \kappa\left(\Delta^b\right) = \theta^g \left[R^g - \tilde{R}^g (1 - \hat{e})\right] - \kappa(\hat{e} - e^* + \delta)
\]
or
\[
\sigma(\hat{e}) = \theta^b \tilde{R}^b \left(e^* - \hat{e} + \Delta^b\right) + \kappa(\hat{e} - e^* + \delta) - \kappa\left(\Delta^b\right)
\]
By the implicit function theorem,
\[
\frac{\partial e^*}{\partial \hat{e}} = 1 - \frac{\sigma'(\hat{e})}{\kappa'(\hat{e} - e + \delta) - \theta^b \tilde{R}^b} = \frac{\kappa'(\hat{e} - e + \delta) - \theta^g \tilde{R}^g}{\kappa'(\hat{e} - e + \delta) - \theta^b \tilde{R}^b} > 0
\]
A.10
which follows from $\hat{e} - e + \delta \geq \Delta^g > \Delta^b$. Therefore, in this region as well, increasing capital requirements worsens legacy zombie lending.

ii. The last case is when $\hat{e}$ is so low that even the bank with $e = \hat{e} - \Delta^g + \delta$ prefers to lend to its legacy $B$ borrower, that is

$$\sigma (\hat{e}) < \kappa (\Delta^g) - \kappa (\Delta^b) + \theta^b \tilde{R}^b (\delta - \Delta^g + \Delta^b)$$  \hspace{1cm} (A.6)

holds, and so all the banks with lower equity also rollover the $B$ loan. Then the indifference threshold $e^*$ is above $\hat{e} - \Delta^g + \delta$ and is the same as in the absence of a capital requirement:

$$e^* = 1 - \frac{\theta^g R^g - \theta^b R^b}{\theta^g \tilde{R}^g - \theta^b \tilde{R}^b} - \frac{\varphi(\theta^g \tilde{R}^g) - \varphi(\theta^b \tilde{R}^b)}{\theta^g \tilde{R}^g - \theta^b \tilde{R}^b} + \frac{\theta^g \tilde{R}^g}{\theta^g \tilde{R}^g - \theta^b \tilde{R}^b} \delta$$

so does not vary with $\hat{e}$. Low enough capital requirements become irrelevant for legacy zombie lending.

For banks matched with a good firm, since we abstract from switching costs $\delta$, they will switch to a new zombie borrower if their post-issuance equity is below

$$E^* = 1 - \frac{\theta^g R^g - \theta^b R^b}{\theta^g \tilde{R}^g - \theta^b \tilde{R}^b}$$

hence capital requirements have a knife-edge effect: either $\hat{e} \leq E^*$ and the capital requirement is irrelevant, or $\hat{e} \geq E^*$ and the capital requirement prevents all these banks (matched with a $G$ firm) from switching to a new $B$ borrower. Since we just showed that increasing $\hat{e}$ can never decrease legacy zombie lending, the only potential benefit is to prevent “new” zombie lending.

Next, note that the point $\hat{e}$ such that (A.6) holds with equality, that is

$$\hat{e} = E^* + \Delta^g - \frac{\varphi(\theta^g \tilde{R}^g) - \varphi(\theta^b \tilde{R}^b)}{\theta^g \tilde{R}^g - \theta^b \tilde{R}^b} + \frac{\theta^b \tilde{R}^b}{\theta^g \tilde{R}^g - \theta^b \tilde{R}^b} \delta$$

is strictly above $E^*$ since $\sigma(\hat{e}) = \kappa(\Delta^g) - \kappa(\Delta^b) + \theta^b \tilde{R}^b (\delta - \Delta^g + \Delta^b) > 0 = \sigma(E^*)$.

The following result generalizes Proposition 2 and characterizes the optimal policy, in the case of quadratic equity issuance costs $\kappa(x) = \frac{1}{a}x^2$ that allow for closed-form solutions:
Proposition 8 (Optimal policy with equity issuance). Output reaches its potential \((Y = Y^*)\) if and only if
\[
\underline{R_f}(p) \leq R_f \leq \bar{R}_f(p)
\]
and
\[
p \leq \hat{p}
\]
where \(\underline{R}_f(p)\) and \(\bar{R}_f(p)\) are given in the Appendix.

The limit case \(a \to 0\) recovers the no-issuance benchmark from Proposition 2. Under quadratic issuance costs, the optimal policy is characterized by the thresholds
\[
\underline{R}_f(p) = p \left( 1 - \frac{\theta^g + \theta^b}{2} \right) - \frac{1}{a} \left( 1 - e_{\text{min}} \right) \left[ \frac{\hat{p}}{p} - 1 \right]
\]
\[
\bar{R}_f(p) = \frac{1}{1 + ap(1 - \theta^g)} \bar{R}_{\text{no issuance}}(p) + \frac{ap^2(1 - \theta^g)^2}{2(1 + ap(1 - \theta^g))}
\]
and \(\hat{p}\) and \(\bar{R}_{\text{no issuance}}(p)\) are as defined in Proposition 2.

Proof of Proposition 8. Banks choose borrower type based on their post-issuance equity \(e' = e + \Delta e\). Define the function \(\varphi(x) = x (\kappa')^{-1}(x) - \kappa \left( (\kappa')^{-1}(x) \right)\). There are two cases to consider:

Case 1. A bank with pre-issuance equity \(e\) prefers lending to a type \(G\) borrower at rate \(R^g\) instead of lending to a type \(B\) borrower if:
\[
\theta^g \left( R^g - \bar{R}^g (1 - e - \Delta^g) \right) - \kappa (\Delta^g) \geq \theta^b \left( R^b - \bar{R}^b (1 - e - \Delta^b) \right) - \kappa (\Delta^b)
\]
which can be rewritten as
\[
e > e^* = 1 - \frac{\theta^g R^g - \theta^b R^b}{\theta^g R^g - \theta^b R^b} - \frac{\varphi \left( \theta^g \bar{R}^g \right) - \varphi \left( \theta^b \bar{R}^b \right)}{\theta^g R^g - \theta^b R^b}.
\]

Case 2. A bank with pre-issuance equity \(e\) prefers investing its capital in safe assets rather than lending to a type \(G\) borrower at rate \(R^g\) if:
\[
R^f \left( e + \Delta^f \right) - \kappa (\Delta^f) \geq \theta^g \left( R^g - \bar{R}^g (1 - e - \Delta^g) \right) - \kappa (\Delta^g)
\]
which can be rewritten as

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\[ e > e^{**} = 1 - \frac{Rf - \varrho Rg}{Rf - \varrho \tilde{R}g} - \frac{\varphi(Rf) - \varphi(\varrho \tilde{R}g)}{Rf - \varrho \tilde{R}g}. \]

D Forward-looking firm dynamics

Incumbent firms draw a cost shock \( \epsilon \) in each period. If they do not exit they earn current expected profit

\[ \pi_i^f(\epsilon) = \theta_i (y_i^t - R_i^t) - c_t - \epsilon \]

Assume firms exit when their project fails. A forward-looking incumbent firm’s value function if it does not exit is

\[
\Pi_i^f(\epsilon) = \pi_i^f(\epsilon) + \beta \theta_i \mathbb{E}_t \left[ (1 - \lambda_i) \max \left\{ \Pi_{i+1}^f(\epsilon_{i+1}), 0 \right\} + \lambda_i \max \left\{ \Pi_{i+1}^{-i}(\epsilon_{i+1}), 0 \right\} \right] = W_{i+1}^i
\]

where with a probability \( \lambda_i \) the firm can change type to \(-i\) next period. Then the firm does not exit if and only if

\[ \Pi_i^f(\epsilon) \geq 0 \iff \epsilon \leq \bar{\epsilon}_i^f = \theta_i (y_i^t + \beta W_{i+1}^i - R_i^t) - c_t \]

A myopic firm ignores the \( W_{i+1}^i \) part, hence does not exit if and only if \( \pi_i^f(\epsilon) \geq 0 \), i.e.,

\[ \epsilon \leq \theta_i (y_i^t - R_i^t) - c_t. \]

Potential entrants are all of the \( i = g \) type, and have cost \( c_t - \gamma - \epsilon \). If they enter they must pay an entry cost \( \kappa \), hence they earn current expected profit

\[ \pi_i^n(\epsilon) = \theta_g (y^g - R_i^g) - c_t + \gamma - \epsilon - \kappa \]

in the first period. After one period they become incumbents and lose their productivity advantage \( \gamma \) (it is straightforward but inconvenient to generalize to \( \gamma \) lasting multiple periods). Thus a potential entrant enters if and only if

\[ \epsilon \leq \bar{\epsilon}_i^n = \epsilon_i^g + \gamma - \kappa \]

Incumbents’ value functions satisfy

\[
\Pi_i^f(\epsilon) = \pi_i^f(\epsilon) + \beta \theta_i \left[ (1 - \lambda_i) \int_0^{\bar{\epsilon}_{i+1}^t} \Pi_{i+1}^f(\epsilon') dH(\epsilon') + \lambda_i \int_0^{\bar{\epsilon}_{i+1}^{-i}} \Pi_{i+1}^{-i}(\epsilon') dH(\epsilon') \right]
\]
Since $\epsilon$ is additive and iid, $\Pi^i_t(\epsilon) = \Pi^i_t(0) - \epsilon$ and by definition (in the case of an interior solution which we will check)

$$\Pi^i_t(0) = \bar{\epsilon}^i_t$$

Thus we need only keep track of the two paths of the two thresholds $\{\bar{\epsilon}^g_t, \bar{\epsilon}^b_t\}$. Rearranging the Bellman equation, they solve

$$\bar{\epsilon}^i_t = \pi^i_t(0) + \beta^i \left[ (1 - \lambda^i) \int_{0}^{\bar{\epsilon}^i_t} (\bar{\epsilon}^{L.o}_{t+1} - \epsilon') dH(\epsilon') + \lambda^i \int_{0}^{\bar{\epsilon}^{-i.o}_{t+1}} (\bar{\epsilon}^{-i.o}_t - \epsilon') dH(\epsilon') \right]$$

If $H$ is uniform between 0 and 1, this simplifies to two quadratic equations.